Part I:
Plane Stress and Plane Strain
The Stress Equilibrium Equation

• As we mentioned in Chapter 2, using the Galerkin formulation and a choice of shape functions, we can derive a discretized form of most differential equations.

• In Structural Mechanics, the governing differential equation will most often be the stress equilibrium equation. We will therefore derive the two dimensional version of it here (see course handouts for the three dimensional version).

• The version we will show is quite standard and was first offered by Cauchy in 1823 by considering the pressure that exists on a plane of arbitrary orientation within a differential element of material (a continuum). In this paper, Cauchy first introduced the notion of a stress tensor.*

*Cauchy was actually not the first person to derive an equilibrium expression for the elastic continuum. This honor usually goes to Navier (1821), but his method was based on a molecular theory of isotropic solids which wasn’t quite correct. For a fascinating account, see:

The Stress Equilibrium Equation

The stress tensor and surface traction

- In the most general sense, any arbitrary continuously varying force, \( \mathbf{F} \) on a surface may be split into a component normal (the pressure) and tangential to the surface.

- In Cauchy’s analysis, this is done on all three sides of the wedge: the inclined side and the two sides of the wedge parallel to the \( x \) and \( y \)-axes, which for the time being are numbered 1 and 2.

- In what follows, we will assume that the force, \( \mathbf{F} \) acts normal to the surface (simply for clarity), so that \( S \) is zero.
The Stress Equilibrium Equation

The stress tensor and surface traction

- The static equilibrium on a “infinitesimal wedge” may be found by considering the pressure on an arbitrary plane.
- The external pressure may be thought of as the component of traction normal to the plane.
- The traction is defined by:

\[ T = \lim_{\Delta A \to 0} \frac{\Delta(F)}{\Delta A} \]

Where \( A \) is the surface area of the plane*, \( F \) is an arbitrary external force acting normal to the plane.

*In our case, this is just \( \Delta L \times 1 = \Delta L \), as we will work on a “per unit thickness basis”
The Stress Equilibrium Equation

The stress tensor and surface traction

\[ \sum F_x = T_x \Delta L - T_2 \Delta L \cos \theta - S_1 \Delta L \sin \theta = 0 \]
\[ T_x = T_2 \cos \theta + S_1 \sin \theta \]

\[ \sum F_y = T_y \Delta L - T_1 \Delta L \sin \theta - S_2 \Delta L \cos \theta = 0 \]
\[ T_y = T_1 \sin \theta + S_2 \cos \theta \]
The Stress Equilibrium Equation

The stress tensor and surface traction

Now, sum moments about point $P$

$$\sum M_p = -S_2\Delta x\Delta y + S_1\Delta y\Delta x = 0$$

$S_2 = S_1$

So, we see that the tangential “pressures” on the two sides parallel to the $x$ and $y$ axes are always equal.

We’ll use this fact to swap the $S_1$ and $S_2$ terms in the previous summation...
The Stress Equilibrium Equation

The stress tensor and surface traction

\[ T_x = T_2 n_x + S_2 n_y \]
\[ T_y = T_1 n_y + S_1 n_x \]

*Or, as a matrix equation*:

\[ \begin{bmatrix} T_x \\ T_y \end{bmatrix} = \begin{bmatrix} T_2 & S_2 \\ S_1 & T_1 \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix} \]  \( 1 \)
The Stress Equilibrium Equation

The stress tensor and surface traction

- The matrix of normal and tangential pressures is known as the Cauchy or infinitesimal stress tensor. This is the same stress tensor you encounter in introductory courses in stress analysis. It’s two-dimensional form is shown below. It gets this name because it is only valid for small increments of stress and corresponding strain.

\[
\begin{bmatrix}
T_2 & S_2 \\
S_1 & T_1
\end{bmatrix}
=\begin{bmatrix}
\sigma_{xx} & \sigma_{xy} \\
\sigma_{yx} & \sigma_{yy}
\end{bmatrix}
\]

- As we saw in slide 5: \( \sigma_{xy} = \sigma_{yx} \) So, the matrix is also symmetric.

- The preceding derivation could have been done in three dimensions following the same procedure, but on a plane cut through a 3D infinitesimal cube (a tetrahedron)...
The Stress Equilibrium Equation

The stress tensor and surface traction

• A diagram like the one below could be used to follow the same procedure as was done in 2D, but in all three dimensions, yielding the full 3 x 3 Cauchy stress tensor

\[
\sigma = \begin{bmatrix}
\sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\
\sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\
\sigma_{zx} & \sigma_{zy} & \sigma_{zz}
\end{bmatrix}
\]

\[
\begin{align*}
\sigma_{xy} &= \sigma_{yx} \\
\sigma_{yz} &= \sigma_{zy} \\
\sigma_{xz} &= \sigma_{zx}
\end{align*}
\]
The Stress Equilibrium Equation

• Next, we’d like to derive the stress equilibrium differential equation itself, which forms the basis for most structural problems.

• Again, we’ll work just in two dimensions to keep things simple (again working in a “per unit-thickness” basis). This time, we’ll use an infinitesimal element with all four sides as shown.

• Here, b is a body force per unit volume.
The Stress Equilibrium Equation

- Applying Newton’s Third Law in the \( x \)-direction gives:

\[
\sum F_x = -\sigma_{xx} \, \Delta y + \sigma_{xx} \, \Delta y - \sigma_{xy} \, \Delta x + \sigma_{xy} \, \Delta x + b \, \Delta x \Delta y = 0
\]

- Rearranging and dividing by \( \Delta x \Delta y \) yields:

\[
\frac{\sigma_{xx} \, \Delta y - \sigma_{xy} \, \Delta x}{\Delta x} + \frac{\sigma_{xy} \, \Delta y - \sigma_{xy} \, \Delta x}{\Delta y} + b_x = 0
\]

- Taking the limit as \( \Delta x \to 0, \Delta y \to 0 \):

\[
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + b_x = 0
\]
The Stress Equilibrium Equation

• Similarly, repeating the previous three steps in the y-direction yields:

\[
\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} + b_y = 0
\]

• And, once again, even though we won’t go thru the steps, we will simply point out that the full three dimensional equations can be obtained in a similar manner, considering a three-dimensional cube element instead of a square. Without showing the derivation, the equations are:

\[
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + b_x = 0
\]

\[
\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + b_y = 0
\]

\[
\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z = 0
\]
The Stress Equilibrium Equation

• Regardless of spatial dimension, the stress equilibrium equation can be written compactly in indicial notation as:

\[ \sigma_{ij,j} + b_i = 0 \]

• Where, as usual, the Einstein summation convention implies summation over repeated indices and the comma in the subscript means “derivative with respect to the coordinate variable implied by the index following the comma”
Small Strains

• The stress equilibrium equations derived previously give us only half the picture. To fully solve for the response of a bounded continuum (a structure), we need a way to relate stresses and forces to strains and displacements, respectively. So, before we look at that, we will introduce the common notion of small, or infinitesimal strains.

• From the work we’ve already done, we know that in a structural problem, we relate forces to displacements through Hooke’s Law (remember it kept coming back again and again). Further, we saw that in reduced continuum solutions, we had solutions over elements (in two dimensions) of the form:

\[ u = f(x, y) = \sum_{i=1}^{n} N_i(x, y)u_i \]

\[ v = g(x, y) = \sum_{i=1}^{n} N_i(x, y)v_i \]
Small Strains

• Just as the displacement field is the natural counterpart to the external forces in a structure, the strain field is the natural counterpart to the stress field in a structure. So, in two spatial dimensions, we can expect four strain components (three independent ones due to the symmetry in slide 7).

• We will designate the strain tensor as $\boldsymbol{\varepsilon}$.

\[
\boldsymbol{\varepsilon} = \begin{bmatrix}
\varepsilon_{xx} & \varepsilon_{xy} \\
\varepsilon_{yx} & \varepsilon_{yy}
\end{bmatrix}
\]

And, in three dimensions:

\[
\boldsymbol{\varepsilon} = \begin{bmatrix}
\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\
\varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\
\varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz}
\end{bmatrix}
\]
Small Strains

Although it’s a little difficult to convey graphically, an infinitesimal element undergoing nonzero $\varepsilon_{xx}, \varepsilon_{yy}$, and $\varepsilon_{yz}$ represents a small perturbation from an equilibrium configuration (the dashed square), such that we can assume the actual reference lengths, $\Delta x$ and $\Delta y$ don’t change (this is a linear deformation assumption). With this in mind, the strain components are defined as:

\[
\varepsilon_{xx} = \lim_{\Delta x \to 0} \left( \frac{u_{x, y + \Delta y} - u_{x, y}}{\Delta x} \right) = \partial u / \partial x
\]

\[
\varepsilon_{yy} = \lim_{\Delta y \to 0} \left( \frac{v_{x + \Delta x, y} - v_{x, y}}{\Delta y} \right) = \partial v / \partial y
\]

\[
\frac{\partial v}{\partial x} = \lim_{\Delta x \to 0} \left( \frac{v_{x + \Delta x, y} - v_{x, y}}{\Delta x} \right) = \tan \theta_x \approx \theta_x
\]
Small Strains

And finally:

\[
\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \left( \frac{u \big|_{x,y+\Delta y} - u \big|_{x,y}}{\Delta y} \right) = \tan \theta_y \approx \theta_y
\]

• The engineering shear strain is given by:

\[
\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}
\]

• The tensor shear strain is \(\frac{1}{2}\) the engineering shear strain:

\[
\varepsilon_{xy} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)
\]
Small Strains

• Thus, it should be clear that

\[ \gamma_{xy} = \gamma_{yx} \quad \text{and} \quad \varepsilon_{xy} = \varepsilon_{yx} \]

• Note that the engineering strain represents the TOTAL shear angle, whereas the tensor strain represents an average.

• Once again, without going thru the details, we’ll point out that in three dimensions, there are three additional unique strain components (letting \( w \) stand for displacement in the \( z \)-direction)

\[
\varepsilon_{zz} = \lim_{\Delta y \to 0} \left( \frac{v|_{x,y,z+\Delta z} - v|_{x,y,z}}{\Delta z} \right) = \frac{\partial w}{\partial z} \\
\varepsilon_{xz} = \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = \frac{1}{2} \gamma_{xz} \\
\varepsilon_{yz} = \frac{1}{2} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = \frac{1}{2} \gamma_{yz}
\]
Generalized Hooke’s Law

• As we mentioned earlier, in order to fully solve the response of a bounded continuum to an external load, we need a constitutive relation between the stress and strain components. This is understood as a property of the continuum (different materials will respond differently over the same bounded continuum under a given loading regime).

• So, we’re looking for something of the form $\sigma = C \varepsilon$, where $C$ plays the role of $K$ in Hooke’s law.

• Unfortunately, a thorough discussion of this mathematical object is beyond the scope of this course*. However, we’ll hit the introductory highlights. We’ll begin by noting that in its most general form, $C$ is a fourth rank tensor, and in indicial notation, the relation between stress and strain is given by:

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl} \quad (3a)$$

*We suggest the interested student begin by looking up “Hooke’s Law” in Wikipedia, for a pretty good introduction.
Generalized Hooke’s Law

• In three spatial dimensions, the tensor $C$ has 81 (3 x 3 x 3 x 3) components! However, if we consider the stress and strain tensors to be symmetric (as we do for small linear small strain problems), the corresponding entries in $C$ will also be symmetric. This double symmetry allows us to reduce the rank of $C$ by two and the rank of $\sigma$ and $\varepsilon$ by 1. This rank-reduction is very convenient for doing calculations and is known as Voigt notation*. Thus, (3a) may be written as:

$$\sigma_i = C_{ij} \varepsilon_j \quad (3b)$$

• Note that with this rank reduction, the 2nd rank stress and strain tensors are replaced with the equivalent stress and strain vectors:

$$\sigma = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix} \quad \varepsilon = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix}$$

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Hooke’s Law for isotropic materials

• This is done to compensate for the loss of a factor of two in the shear terms.

Note in particular that the tensor shear strain components are replaced by engineering shear strains! when expressing the internal strain energy density...

\[ V = \frac{1}{2} \sigma_{ij} \varepsilon_{ji} \]  \hspace{1cm} (4)

...in the equivalent vector form:

\[ V = \frac{1}{2} \sigma^T \varepsilon \]

• So, we see that in three spatial dimensions, \( C \) has 36 unique terms due to the symmetry of the stress and strain tensors \((6 \times 6)\) terms. If the material property is also symmetric about \( x, y, \) and \( z \), then the 36 unique terms reduce to 21 independent terms. If we further require that the continuum is isotropic, then we’re left with only two independent variables!

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
\sigma_{xy} \\
\sigma_{xz} \\
\sigma_{yz}
\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}
\begin{bmatrix}
1-\nu & \nu & \nu & 0 & 0 & 0 \\
\nu & 1-\nu & \nu & 0 & 0 & 0 \\
\nu & \nu & 1-\nu & 0 & 0 & 0 \\
0 & 0 & 0 & (1-2\nu)/2 & 0 & 0 \\
0 & 0 & 0 & 0 & (1-2\nu)/2 & 0 \\
0 & 0 & 0 & 0 & 0 & (1-2\nu)/2
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\varepsilon_{zz} \\
\gamma_{xy} \\
\gamma_{xz} \\
\gamma_{yz}
\end{bmatrix} \hspace{1cm} (5)
\]
Hooke’s Law for isotropic materials

• So, equation (5) gives us the constitutive relation for an isotropic material in three dimensions, such that $C$ in equation (3b) is:

$$C = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix}
1-\nu & \nu & \nu & 0 & 0 & 0 \\
\nu & 1-\nu & \nu & 0 & 0 & 0 \\
\nu & \nu & 1-\nu & 0 & 0 & 0 \\
0 & 0 & 0 & (1-2\nu)/2 & 0 & 0 \\
0 & 0 & 0 & 0 & (1-2\nu)/2 & 0 \\
0 & 0 & 0 & 0 & 0 & (1-2\nu)/2 \\
\end{bmatrix}$$

• If we had some other type of material (perhaps orthotropic, or transversely isotropic for example), we would simply replace $C$ with the appropriate constitutive relation. Any other material will have more non-zero terms and more independent variables. The minimum number of independent (scalar) material properties for a continuum in at least two spatial dimensions is two.

• We have given equation (5) in a particular form, where $E$ is the Elastic Modulus and $\nu$ is Poisson’s Ratio
The Elastic Modulus and Poisson’s Ratio

• The elastic modulus, $E$ (also known as Young’s Modulus) measures the material’s tensile strain under a uniform tensile stress, while Poisson’s Ratio gives a measure of how much the material contracts in a direction transverse to the uniform tensile stress (this is known as Poisson’s Effect). This may be seen by taking the inverse of equation (5):

$$
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\varepsilon_{zz} \\
\gamma_{xy} \\
\gamma_{yz} \\
\gamma_{xz}
\end{bmatrix} = \frac{1}{E} \begin{bmatrix}
1 & -\nu & -\nu & 0 & 0 & 0 \\
-\nu & 1 & -\nu & 0 & 0 & 0 \\
-\nu & -\nu & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\
0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\
0 & 0 & 0 & 0 & 0 & 2(1+\nu)
\end{bmatrix} \begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
\sigma_{xy} \\
\sigma_{yz} \\
\sigma_{xz}
\end{bmatrix}
$$

(6)

• Now, apply a uniaxial stress, $\sigma = \begin{bmatrix} \sigma_{xx} & 0 & 0 & 0 & 0 \end{bmatrix}^T$
The Elastic Modulus and Poisson’s Ratio

• This results in the equations:

\[ \varepsilon_{xx} = \frac{1}{E} \sigma_{xx} \quad (7a) \]
\[ \varepsilon_{yy} = -\frac{\nu}{E} \sigma_{xx} \quad (7b) \]
\[ \varepsilon_{zz} = -\frac{\nu}{E} \sigma_{xx} \quad (7c) \]

• The fundamental linear relationship between stress and strain is embodied in (7a), and we may in fact define the elastic modulus \( E \) under a uniaxial stress, \( \sigma_{xx} \) from (7a) as:

\[ E = \frac{\sigma_{xx}}{\varepsilon_{xx}} \quad (8) \]

• Furthermore, (7b) and (7c) state that there are two transverse components of compressive strain (the sign of the strain is opposite that of the stress) resulting from the uniaxial stress field. And, if we substitute \( \sigma_{xx} = E\varepsilon_{xx} \) into (7b) and (7c) to solve for \( \nu \), we obtain:

\[ \nu = -\frac{\varepsilon_{yy}}{\varepsilon_{xx}} = -\frac{\varepsilon_{zz}}{\varepsilon_{xx}} \quad (9) \]
The Elastic Modulus and Poisson’s Ratio

• In other words, Poisson’s ratio may be thought of as a ratio of transverse to axial strain (or contraction) under a uniaxial stress as shown in the figure below. It is worth repeating: This a material property. Different materials will exhibit different ratios of contraction under a given tensile load (the values may even be negative or zero, but they must never exceed 0.5, in which case the material is perfectly incompressible):
Plane Stress

• When analyzing bounded continua in two spatial dimensions, one can make one of two assumptions. The first one is that all stress components in the third dimension (let’s call it z) are zero. In other words:

\[
\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0
\]

• We plug this assumption into (6):

\[
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\varepsilon_{zz} \\
\gamma_{xy} \\
\gamma_{yz} \\
\gamma_{xz}
\end{bmatrix}
= \frac{1}{E}
\begin{bmatrix}
1 & -\nu & -\nu & 0 & 0 & 0 \\
-\nu & 1 & -\nu & 0 & 0 & 0 \\
-\nu & -\nu & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\
0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\
0 & 0 & 0 & 0 & 0 & 2(1+\nu)
\end{bmatrix}
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
\sigma_{xy} \\
\sigma_{yz} \\
\sigma_{xz}
\end{bmatrix}
\]

These components are zero

to yield:

\[
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\gamma_{xy}
\end{bmatrix}
= \frac{1}{E}
\begin{bmatrix}
\sigma_{xx} - \nu \sigma_{yy} \\
-\nu \sigma_{xx} + \sigma_{yy} \\
2(1+\nu) \sigma_{xy}
\end{bmatrix}
\]

(10)

\[
\varepsilon_{zz} = -\frac{\nu}{E} \left( \sigma_{xx} + \sigma_{yy} \right)
\]

(11)
Plane Stress

• Note that separation of the z-component strain reaction (due purely to Poisson’s Effect) in (11) is integral to the plane stress assumption. We omit it from the constitutive relation and apply it as a bookkeeping exercise after calculating the in-plane reactions.

• Re-writing (10) as a matrix equation:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}$$

(12)

• And, after inversion (using Mathematica®):

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \end{bmatrix}$$

(13)

And remember our bookkeeping:

$$\varepsilon_{zz} = -\frac{\nu}{E} (\sigma_{xx} + \sigma_{yy})$$
Plane Stress

• Note that inverting a truncated (2-dimensional) version of (6) did NOT yield a corresponding truncated version of (5)! This is very important and distinguishes plane stress from plane strain from a purely mathematical standpoint. It also reveals that the plane stress assumption is approximate, whereas the plane strain condition can be applied to a real system exactly (we will repeat this later)!

• The assumptions of plane stress are an approximation. In real structures, the out-of-plane (z) strain would in fact be accompanied by a corresponding out-of-plane stress as required by (5). However, the approximation is good under certain circumstances which we will discuss later.
Plane Strain

• The other 2D assumption we could make is that all the strain components in the z-direction are zero:

\[ \varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0 \]

• This is a different kind of assumption because it can be explicitly enforced (it may be stated as a boundary condition on the LHS of the governing differential equation), whereas the plane stress assumption arises from an observed approximation due to the absence of loads, gradients, and boundary conditions in the z-direction.

• Simply plug the above assumptions into (5):

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
\sigma_{xy} \\
\sigma_{xz} \\
\sigma_{yz}
\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix}
1-\nu & \nu & \nu & 0 & 0 & 0 \\
\nu & 1-\nu & \nu & 0 & 0 & 0 \\
\nu & \nu & 1-\nu & 0 & 0 & 0 \\
0 & 0 & 0 & (1-2\nu)/2 & 0 & 0 \\
0 & 0 & 0 & 0 & (1-2\nu)/2 & 0 \\
0 & 0 & 0 & 0 & 0 & (1-2\nu)/2
\end{bmatrix} \begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\varepsilon_{zz} \\
\gamma_{xy} \\
\gamma_{xz} \\
\gamma_{yz}
\end{bmatrix}
\]
Plane Strain

• This yields:

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{yx}
\end{bmatrix} = \frac{1}{(1+\nu)(1-2\nu)} \begin{bmatrix}
1-\nu & \nu & 0 \\
\nu & 1-\nu & 0 \\
0 & 0 & (1-2\nu)/2
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\gamma_{xy}
\end{bmatrix}
\]

(14)

• And:

\[
\sigma_{zz} = \frac{Ev}{(1+\nu)(1-2\nu)} (\varepsilon_{xx} + \varepsilon_{yy})
\]

Where we again remove the out-of-plane reaction purely due to Poisson’s Effect and calculate it separately.

• As we mentioned before, this condition can be explicitly enforced in a structure. Specifically, if we fix the z-displacements at the extreme z-locations (the “ends” of the structure) and ensure no load gradients in the z-direction, then the plane strain assumption should hold at locations far from the ends.
Plane Strain (axisymmetric version)

• The planar axisymmetric formulation is another planar stress/strain formulation, but we do not include it as a separate category because it is actually equivalent to the plane strain formulation in cylindrical coordinates (all properties of plane strain hold equally true of axisymmetric elements). Switching to a cylindrical reference frame does introduce some interesting differences, however.

...all that is required to convert the plane strain formulation to an axisymmetric one is replace the y-coordinate with the cylindrical z, the x-coordinate with r, and z coordinate with $\theta$. This produces a 4 x 4 constitutive matrix and 4 x 1 stress/strain vector:
Plane Strain (axisymmetric version)

\[
\begin{bmatrix}
\sigma_r \\
\sigma_\theta \\
\sigma_z \\
\sigma_{rz}
\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix}
1-\nu & \nu & \nu & 0 \\
\nu & 1-\nu & \nu & 0 \\
\nu & \nu & 1-\nu & 0 \\
0 & 0 & 0 & 1/2-\nu
\end{bmatrix} \begin{bmatrix}
\varepsilon_r \\
\varepsilon_\theta \\
\varepsilon_z \\
\gamma_{rz}
\end{bmatrix}
\]
Part II: Plane Stress and Plane Strain (differences, singularities, and when to apply each)
We’d like to know *when* we can reduce a 3D problem to a 2D problem. To make things a little simpler, we’ll introduce the planar continuum condition: If a model problem satisfies the following two conditions, we should be able to reduce the problem to 2D (but not yet select a particular 2D formulation):

1. The cross section normal to a given direction is constant along that direction
2. The model loads, material properties, and boundary conditions are constant along the same direction

As we’ll see shortly, these properties are necessary, but not sufficient to justify using plane stress, plane strain, or axisymmetric elements.

In part I, we learned that in a plane stress problem, all z-components of stress are zero. In a plane strain problem, all z-components of strain are zero. But what does this “look like”? What physical circumstances produce these properties (or how can we enforce them)?
Example 1

• For the first example, consider the model given below

Green arrows denote applied displacements, while red arrows denote applied pressure

• This model obviously satisfies the conditions for a planar continuum given previously. Let us additionally apply Cartesian planar symmetry boundary conditions on the planes shown
Example 1

Now, the crucial question to answer is: “What happens on the free face?”
Example 1

• When we perform the analysis, we find the following

• There is a maximum z-component strain at this corner on the free end

• There is a maximum von Mises stress at a location very close to the same corner

• These maxima are highly localized and represent singularities associated with a sharp corner at the free end!
This is a good time to introduce the idea of stress/strain singularities in structural mechanics. They are a constant recurring problem for the analyst and it is vital that the student understand when they occur and why.

We will try to give a simple, engineering definition: A stress/strain singularity occurs in a model whenever a material condition (which can involve either the geometry, boundary condition, or material property) or load distribution changes abruptly – one with respect to the other.

We might also think of an abrupt change in material condition as being an abrupt change in the resisting medium with respect to the load. When this occurs with a boundary condition, we must add a caviat: Any abrupt change in boundary condition apart from the minimal constrained DoF set required to prevent rigid-body motion (so predicting a singularity may not be trivial).

By “abrupt”, we mean that the change cannot be represented by a single differentiable function.
• Because either the load or resisting medium changes abruptly (over a span of zero area or length), the derivatives of the solution may be expected to be unbounded at such locations (tend to infinity).

• In practice, this means that the finite element solution *may* show very high stresses and strains at these locations, which only increase with increasing mesh resolution. However, a singularity need not necessarily invoke a high peak stress – it is just a point of instability. Below are some examples of stress singularities.

In a 2D and 3D continuum, there are singularities at these three locations.
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Plane Stress/Strain and Singularities

• Some more examples...

There is a singularity at this corner.

And here...

Symm. y

Symm. x

Symm. y

Symm. x

Symm. y

Symm. x
Example 1

So, back to example 1. Let’s see what happens when we vary the length of the model. We track the maximum strain and stress along the z-axis (the length) on the symmetry (XY) face.

Max. z-component stress and strain on the symmetry face.
Example 1

- It might also be instructive to graph the von Mises stress on the symmetry face as the length changes:

Max. von Mises stress on the symmetry face
Example 1

• The maximum z-component stress seems to be converging to a value of around 75 ksi, while the maximum z-component strain is heading for zero. Meanwhile, the von Mises stress is leveling out to around 247 ksi

• We might deduce from this that the reaction at the symmetry face levels off with increasing part length due it’s increasing distance from the singularity at the free face (if true, this would be an illustration of Saint-Venant’s Principle*)

• But the story gets more interesting...

* http://en.wikipedia.org/wiki/Saint-Venant%27s_principle
Example 2

Let’s now see what reactions we get to the same symmetry face when we additionally impose a symmetry boundary condition to the formerly free face...

Max. z-component stress and strain on the symmetry face
Example 2

...and again we look at the von Mises stress

![Graph showing von Mises stress values]

- 249596.5
- 247749.7
Example 2

• So let’s review what we see in example 2: The maximum z-component stress is relatively constant at around 75.4 ksi. The maximum z-component strain is numerically zero, while the maximum von Mises stress is also relatively constant at around 248 ksi. That the strain is zero should not surprise us.

• The imposition of two parallel symmetry planes effectively forces the z-component strain to zero (because there are no gradients in z) while simultaneously making the part act identical to one with infinite length (on the right).
Preliminary Conclusions

• In any case, the zero z-component strain (we haven’t checked the other z-component strains, but they’re zero too), coupled with the planar stress/strain conditions mean that example 2 is effectively a plane strain condition!

• What about example 1? Well, it starts out with a substantially nonzero z-component strain, but relatively small z-component stress. As the length grows, it seems to converge to the results of example 2 (which we’ve already determined is a plane strain condition).

• We might hypothesize that at small lengths, example 1 approximates a plane stress condition. It then tends toward the plane strain condition as the length increases.
Preliminary Conclusions

- Let’s check our guesses by doing a comparison. First we’ll look at the plane stress hypothesis by looking at the 3D solution of a thin slice fixed in z only at one face.

.1 thick slice.
Only \( z=0 \) fixed in z-direction.

Max. Seqv = 286.6 ksi
Max. Sz = 38 ksi
Max. ez = 0.003
Preliminary Conclusions

...and the actual plane stress solution:

Max. Seqv=303.9 ksi
Max. Sz = 0 ksi
Max. ez= 0.004
Preliminary Conclusions

• Now let’s check plan strain hypothesis by fixing both faces in z

.1 thick slice.
z=0 AND z=0.1 fixed in z-direction

Max. Seqv=249.6 ksi
Max. Sz = 75.3 ksi
Max. ez= 1.3e-12
Preliminary Conclusions

...and the actual plane strain result...

Max. Seqv=242.9 ksi
Max. Sz = 75.6 ksi
Max. ez= 0.0
Preliminary Conclusions
Let’s summarize:

**Plane stress hypothesis (3D model fixed at only one face)**

<table>
<thead>
<tr>
<th>thickness</th>
<th>z-constraint</th>
<th>Seqv (ksi)</th>
<th>Sz (ksi)</th>
<th>Ez</th>
</tr>
</thead>
<tbody>
<tr>
<td>1”</td>
<td>1 face only</td>
<td>286.6</td>
<td>38</td>
<td>0.003</td>
</tr>
</tbody>
</table>

**Plane stress result (2D model)**

<table>
<thead>
<tr>
<th>Plane stress</th>
<th>Seqv (ksi)</th>
<th>Sz (ksi)</th>
<th>Ez</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>303.9</td>
<td>0</td>
<td>0.004</td>
</tr>
</tbody>
</table>

**Plane strain hypothesis (3D model fixed at both faces)**

<table>
<thead>
<tr>
<th>thickness</th>
<th>z-constraint</th>
<th>Seqv (ksi)</th>
<th>Sz (ksi)</th>
<th>Ez</th>
</tr>
</thead>
<tbody>
<tr>
<td>1”</td>
<td>both faces</td>
<td>249.6</td>
<td>75.3</td>
<td>1.3E-12</td>
</tr>
</tbody>
</table>

**Plane strain result (2D model)**

<table>
<thead>
<tr>
<th>Plane stress</th>
<th>Seqv (ksi)</th>
<th>Sz (ksi)</th>
<th>Ez</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>242.9</td>
<td>75.6</td>
<td>0</td>
</tr>
</tbody>
</table>

The agreement is pretty good. It’s starting to look like our hypotheses are correct...
Preliminary Conclusions

We can summarize our conclusions in a plane stress/strain decision tree as follows:

- **Planar Continuum Problem**
  - **Large thickness**
    - One end free: Plane strain
    - Both ends fixed in z: Plane strain
  - **Small thickness**
    - One end free: Plane stress
    - Both ends fixed in z: Plane strain

We can use this tree for future studies to determine whether to apply plane stress or plane strain (when a model can be reduced to 2D).
More evidence

• So far, we have strong supporting evidence for the preliminary conclusions we drew from the evidence we got from examples 1 and 2. Let’s summarize these conclusions.

• When the planar continuum conditions are met and the thickness is small (compared with the smallest planar dimension, say), with one planar face free, the solution tends to the plane stress condition. If the two end faces are constrained in the normal direction, the solution tends to the plane strain solution regardless of thickness.

• When the planar stress/strain conditions are met and the thickness is large with one planar face free, the solution at parallel planes sufficiently far from the free face tends to the plane strain condition.

• At this point, it might be instructive to ask what mechanism produces this result.
Example 3

• To answer this last question, we will investigate a model problem with no singularity.
Example 3

• As before, we’ll track the maximum z-component stress and strain for different values of L

One face free

Both faces fixed
Example 3

*And when we look at the maximum in-plane stresses (radial and circumferential), the two cases are identical*

Hoop Stress: One face free

Hoop Stress: Both faces fixed
Example 3

**Radial stress:** One face free

**Radial stress:** Both faces fixed
Example 3

• The only difference is, of course in the axial direction.

When we look at axial strain:

Axial strain: One face free

Axial strain: Both faces fixed
Example 3

...and when we look at axial stress:

Axial stress: One face free

Axial stress: Both faces fixed
Final Conclusions

• Example 3 seems to illustrate that in the absence of singularity associated with the free end, the plane stress and plane strain solutions are identical *in plane* – the only difference being that the plane stress solution produces a nonzero axial strain accompanied by a zero axial stress while the plane strain solution has a nonzero axial stress accompanied by a zero axial strain.

• This is further evidence that the plane stress solution is an approximation associated with a free end condition. If the thickness of the part is thin enough, this end condition dominates (leading to the plane stress condition). With sufficient thickness, parallel planar sections far enough from this condition will not be influenced by it at all due to Saint-Venant’s Principle (leading to the plane strain condition).

• We conclude by noting that even in thick parts satisfying the planar stress/strain requirements, a free end with a singularity will have a solution lying somewhere between the plane stress and plane strain conditions.