Static Equivalent Model Reduction in Ansys

Revision 2 10/22/2024:

1. Fixed typos on slides 4- 7 (L(1-L) corrected to L-l)
2. Equation (5) of slide 6 corrected to express the same solution as Equation (6): $\delta = -\frac{l^2(L-l)^2 w}{24EI}$.
   The original mistake was the result of a sign error
3. Corrected mis-statements about the resulting incorrect Equation (5).
4. Added slide 7 to address the case of differing materials.
5. Added summary table of results on slide 31
6. Added an Appendix A to justify slide 7
   Added Appendix B for additional verification (Roark's Formula lookup)
7. Added slides 29 and 30 for additional verification of theoretical moment and shear estimates for solution 3 (moments and shear instead obtained from a new macro provided for model 2)

Important: The original version of this post incorrectly estimated the isolated cantilever (Equation (5) of slide 5) deflection using shear and moment estimtes (Equations (1) and (2)). This should always match the fixed-fixed beam solution of (Equation (4)). Comments that accompanied this error were also incorrect and corrected here. Our many thanks for **Bekele Atnafu** of Honeywell from alerting us to these errors. We'd also like to extend our thanks to **Pablo Alvarez** of Sol Aero for suggesting using Roark's Formulas for additional verificiation –also now included in this revision

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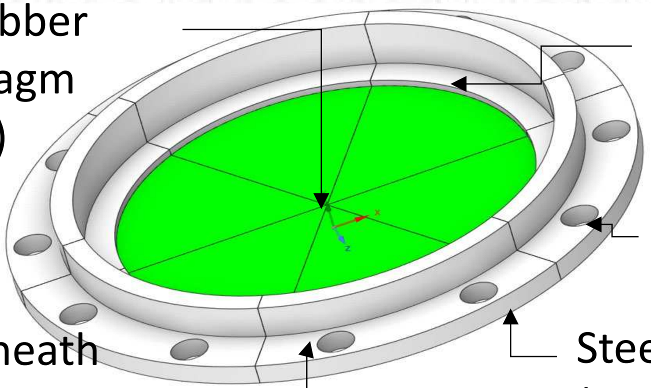
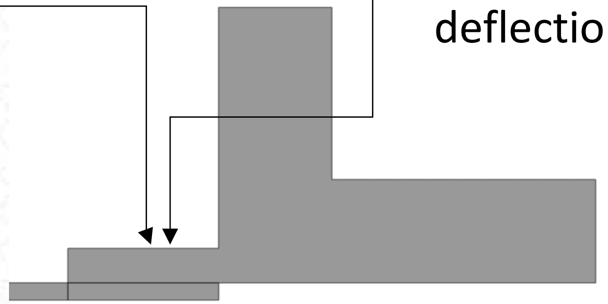
Background

- Ansys users are often tasked with calculating a complex structural system's response at a particular location which happens to be simpler in general than the rest of the model
- When this happens, it is tempting to perform hand-calculations –or at a minimum, to build a simpler model of the region in question subject to an equivalent load
- While straightforward in principle, the technical support crew at PADT have discovered over the years that customers often either encounter subtleties in the static equilibrium calculations –or the implentation of the resulting reduced model in Ansys
- Recently, a customer presented PADT with a model similar to that shown below



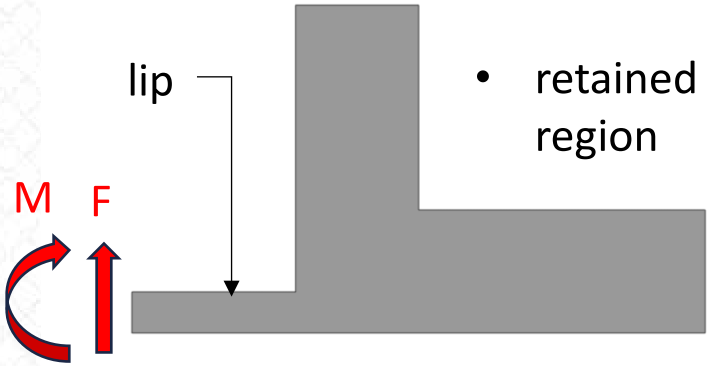


- The customer was interested here in the deflection (and stress) only in the seal attachment (which we henceforth call the 'lip')

Preliminaries

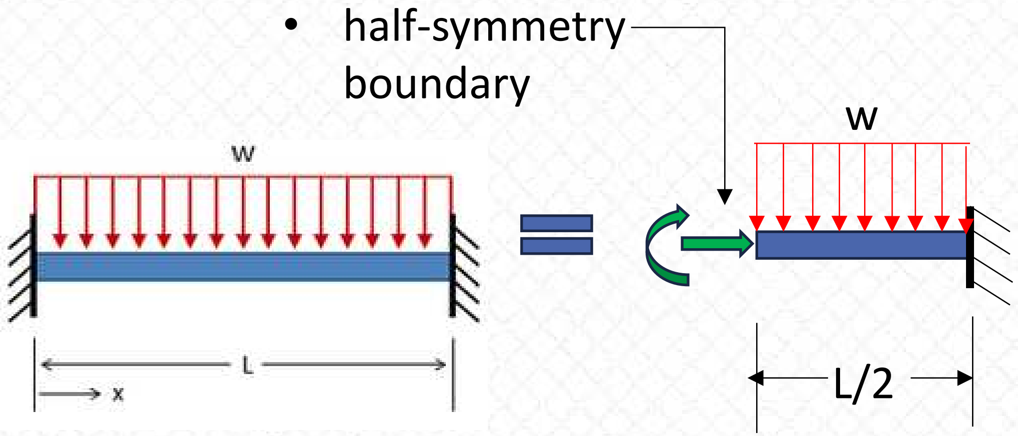
- Since the customer was interested only the deformation and stress of the lipt, his question was: Can he model just the metal steel retainer with statically equivalent loads (resolving the constant pressure underneath the diaphragm into statically equivalent loads on the steel retainer as shown)

- Before answering this question, we should first ask if this is possible in general. If so, what are the restrictions on such a model simplification?



- We might start by considering a simple fixed-fixed beam as below

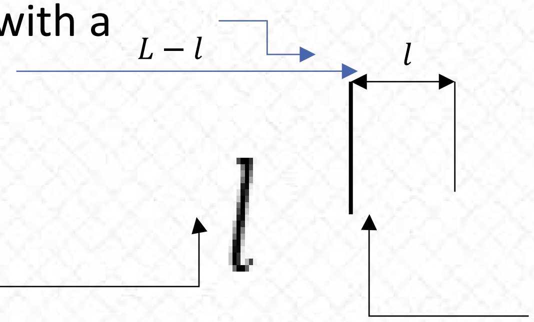
- We know we can impose symmetry and only model half of it...



Preliminaries

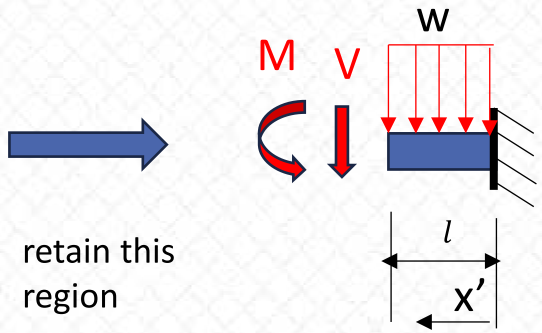
- ...but what we're after is much more ambitious. We're not simply utilizing symmetry to simplify a model
- We're trying to model a geometrically arbitrary (but not *mechanically* arbitrary) section of it as shown below. Can we do this...?

- Is it possible to just model this one truncated section (with a cantilever, say)?



- exclude this region
- retain this region

- Model just the region $L-l \leq x \leq L$ while imposing loads from region $0 \leq x \leq L-l$



- Note that since we have the full analytical solution for a fixed-fixed beam, such a simplification isn't necessary
- We're going through this exercise just to see 'if it works' in more complicated situations. Many readers will already know the answer, but it's worth going through the entire exercise to see

- What we want to do is apply the load contributions from the section we're not modeling to the section we are modeling

Preliminaries

- Since we're keeping section from $\mathrm{L}-l \leq x \leq L$, we can substitute $x=L-l$ into the analytical expressions for for the transverse shear force and moment (which range from 0 to x=L from the left. Readers can find the expressions here)

$$M=w\left(6Lx-6x^2-L^2\right)/12 \quad (1)$$

- The moment solution for a fixed-fixed beam

$$V=w\left(\frac{L}{2}-x\right) \quad (2)$$

- The Shear force expression for a fixed-fixed beam

$$\delta=-\frac{wx^2}{24EI}(L-x)^2 \quad (3)$$

- The transverse deflection of the fixed-fixed beam at x

$$\delta=\frac{Fl^3}{3EI}+\frac{Ml^2}{2EI}-\frac{wl^4}{8EI} \quad (4)$$

- Transverse deflection of the cantilever of length l due to a combination of uniform distributed load w, force F and moment M (the sign of the moment and F come from (1) and (2), while w is considered positive downward to obtain a negative deflection)

Preliminaries

- Substituting in the values for M and F=V and x at $x = L - l$ into (4) yields:

  - the cantilever solution: $$\delta = -\frac{l^2(L-l)^2 w}{24EI} \qquad (5)$$

- Now, compare the solution (4) to the full fixed-fixed beam solution (3) at $x = L - l$:

  - the full solution @ $x = L - l$ $$\delta = -\frac{l^2(L-l)^2 w}{24EI} \qquad (6)$$

- These are the same solutions!
- This problem was chosen as it's the simplest satically indeterminate system we could think of
- The reason this works is that the cantilever solution (eq. 4) incorporates the Shear and bending moment calculations (equations (1) and (2)) used in the full fixed-fixed beam solution
- In other words, we are effectively calculating a 'force-based' submodel

Prerequisites for Static Equivalent Model Reduction

- The previous simple beam example demonstrates that we can discard portions of a model we're not interested in IF we can characterize those portions as boundary conditions (with all degrees of freedom at such boundaries accounted for)
- If some stiffness components of the portion of the model we wish to exclude are negligible compared with the stiffness of the retained portion, we can relax the strict requirements above somewhat.
- For example: if the beam section $0 \leq x \leq L - l$ is made of rubber, while the beam section $L - l \leq x \leq L$ is made of steel, the first section could be thought of simply transmitting a transverse load to the latter section
- As an illustration: if all regions of the model may be characterized by a linear constitutive law, and if the elastic modulus of the excluded region << than that of the retained region, we may approximate the shear and moment (1) and (2) as:

$$V = w\left(\frac{L}{2} - x\right) \approx -\frac{w(L-l)}{2}$$

$$M = \frac{w\left(6Lx - 6x^2 - L^2\right)}{12} \approx -\frac{w(L-l)^2}{12}$$

Preliminaries
Prerequisites for Static Equivalent Model Reduction

- Some readers may wonder when an approximation of the type made on the previous slide can be made.
- To provide more guidance, we offer a detailed derivation of a fixed-fixed beam system modeled by coupling two cantilevers using Lagrange Multipliers in the Appendix
- Even without consulting this solution, however, its quite clear that when all regions have the same elastic modulus, this appoximation results in a shear that is 33% higher than the true value. The moment, however, is a factor of 16 too high (these numbers are the reciprocals of F/F_2 and M/M_2 reported in the Appendix, respectively for the case $E_1 = E_2$).
- So, there are essentially three prerequisites here for excluding a region with a different material than the retained region:

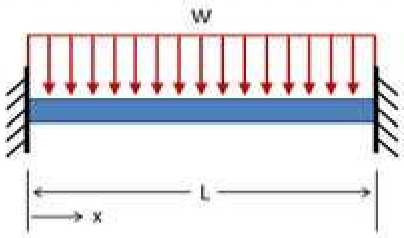
   1. The stiffness components, ke of the excluded region of the model associated with unaccounted degrees of freedom, must be much less than the stiffness of the retained region, kr (ke << kr) in order to be negligible while still transmitting loads (assuming we can't resort to a closed form solution encompassing both excluded and retained regions)
   2. The load path of the full model must be known and faithfully reproduced in the reduced model
   3. The retained model for which we calculate statically equivalent loads must behave linearly (which is not a requirement for the excluded region if we can solve its differential equation)

Common Mistakes

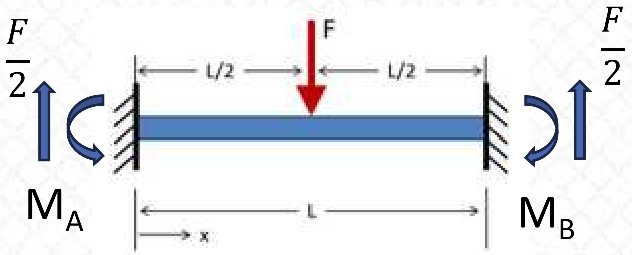
- Before continuing, we should address some common mistakes engineers make when calculating force and moment equilibrium in engineering structures, so we'll be careful to avoid these mistakes going forward

- Calculating moment reactions for statically indeterminate systems
  - Because a distributed load may be represented by an equivalent force acting at the distribution centroid, its useful to reduce a structure with constant distributed load to one with an equivalent concentrated load
  - But see what happens when you do this to a fixed-fixed beam...

[1]



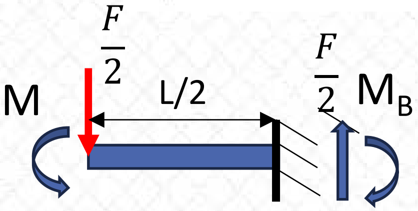
[2]



$$F = wL$$

$$\sum_{M_{L/2}} M - \frac{wL^2}{4} - M_B = 0 \quad (7)$$

- Considering half-symmetry:



- But since $M=M_b$:

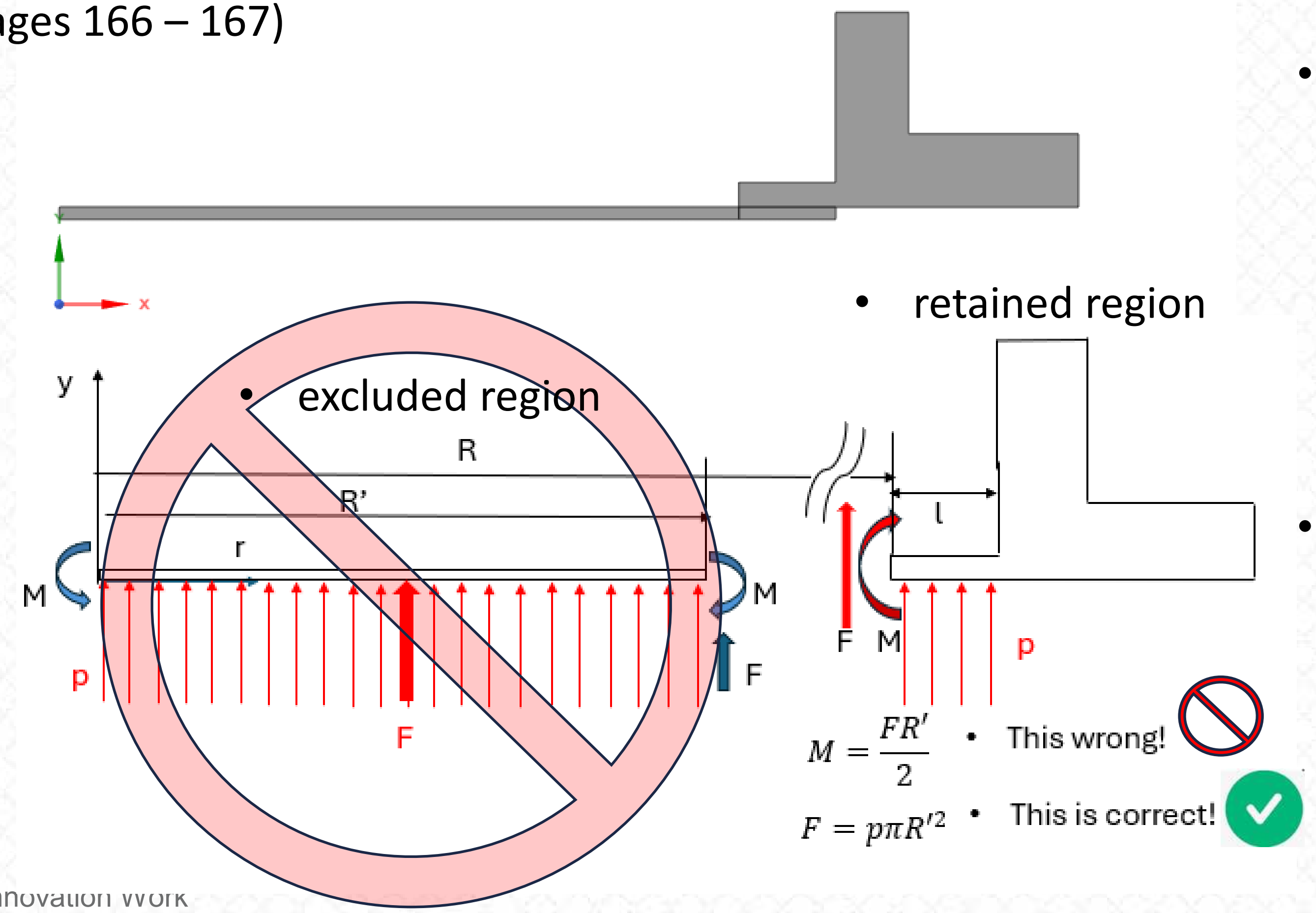
$$-M_A = M_B = \frac{-wL^2}{8} \quad (8)$$

Common Mistakes

- Calculating moment reactions for statically indeterminate systems
  - Equation (8) is wrong! Furthermore, the calculated midspan moment magnitude $wL^2/4$ in equation (7) is wrong, and M cannot be solved from M_B. Both errors are due to the fact that this problem is statically indeterminate!
  - We could summarize the fundamental mistakes as follows:
    1. The shear/moment reactions of the beam with the uniform distributed load is not the same as the one with the concentrated point load (you can quckly compare the two solutions here), so the move from [1] to [2] is wrong, but only because the problem is statically indeterminate. For determinate systems, this is a perfectly valid move.
    2. Equating M and M_b (as pointed out above). Interestingly, although this is incorrect, it produces the correct moment magnitude at the boundaries (For [2]. Not for [1]. Even if the directions are wrong)
  - The solution to this problem is to calculate shear and moments at boundaries using the appropriate beam-deflection solution (for a fixed-fixed beam under distributed load in this case). It is precisely the solutions for a fixed-fixed beam which tell us the solutions of the previous slide are wrong
- Treating an axisymmetric shell like a beam
  - We will be calculating forces and moments for the retained portion of the diaphragm model (see slides 2 and 3) –which happens to be an axisymmetric membrane or plate

Common Mistakes

- Treating an axisymmetric shell like a beam
  - Because of its simplicity, it is tempting to treat the axisymmetric diaphagm as a beam problem
  - This is wrong! A correct analytical expression of the moment can only be found in the solution of a clamped axisymmetric plate under uniform pressure load (assuming it carries moments. See pages 166 – 167)

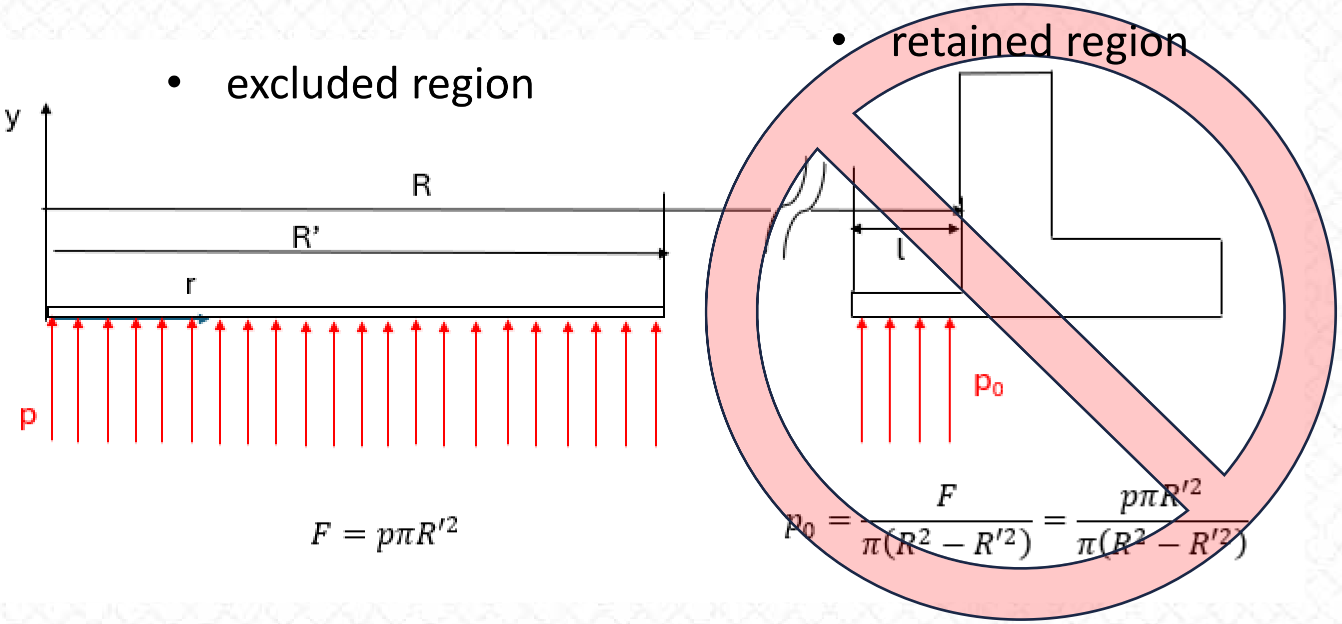


- Even if the diaphragm structure could be analyzed as a beam model (in which case, it would be a clamped roller-fixed beam), we still have the same problem of static indeterminacy as we face with the fixed-fixed beam
- Just as the solution there was to turn to Euler-Bernoulli beam theory, the solution here is to turn to plate theory*

*beams simply don't work here. Consider: How would one characterize the area moment of inertia (what value would it take on at r=0)?

Common Mistakes

- Considering only force equilibrium (while neglecting moment equilibrium)
  - Some engineers will be tempted to believe that they can simply account for the excluded region (the working diaphragm) by accounting only for its axial force over the remaining diaphragm attachment region with a scaled pressure over that region



$$F = p\pi R'^2$$

$$p_0 = \frac{F}{\pi(R^2 - R'^2)} = \frac{p\pi R'^2}{\pi(R^2 - R'^2)} \quad (9)$$

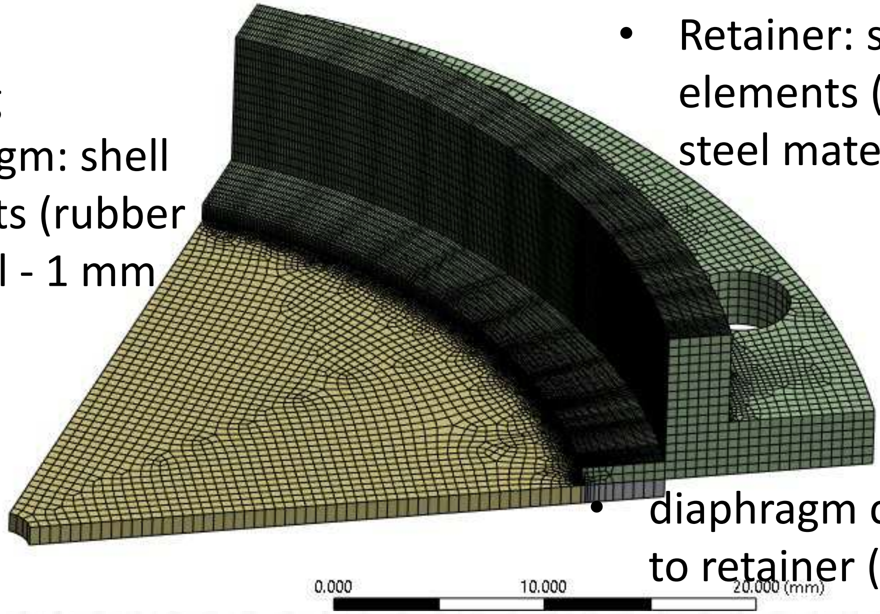
  - This will underpredict the response of the retained structure by neglecting the moment contribution of the excluded region

The Model

- Rubber Diaphragm

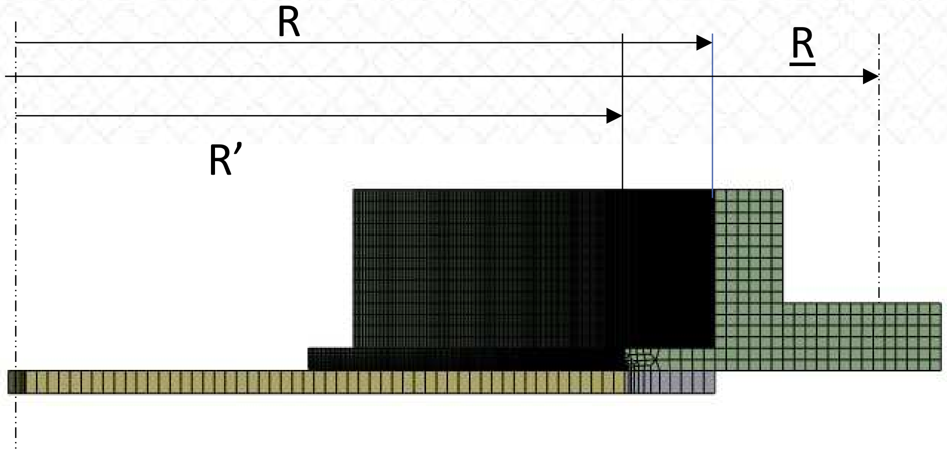
- We'll first build a 1/6-symmetry (60°) finite element model of the structure described on slide 2
- We use shell elements for the diaphragm and solid elements for the retainer
- We use a 3-parameter Mooney-Rivlin material model for the rubber, and the default linear properties of structural steel

- working diaphragm: shell elements (rubber material - 1 mm thick)



- Retainer: solid elements (structural steel material)

- diaphragm connects directly to retainer (no contact)

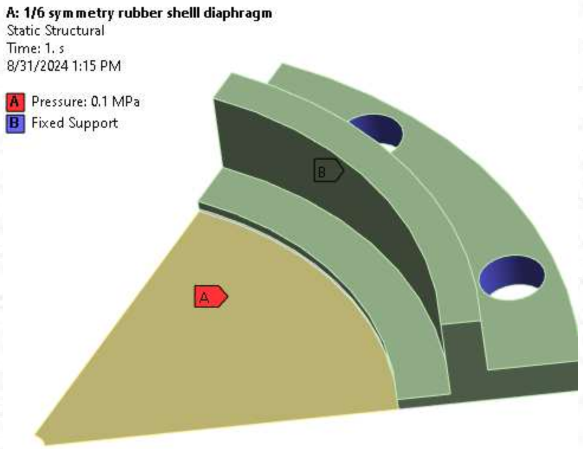


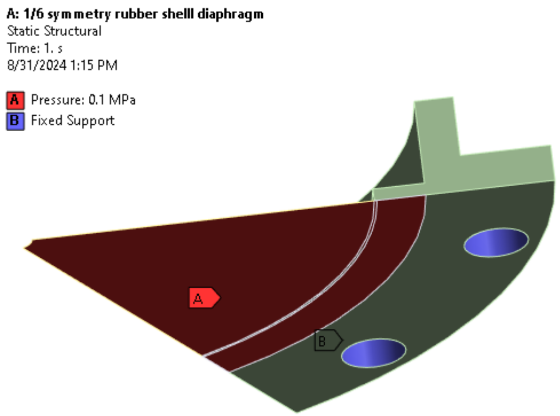
R' = 28.2 mm
R= 32 mm
R = 38.5 mm

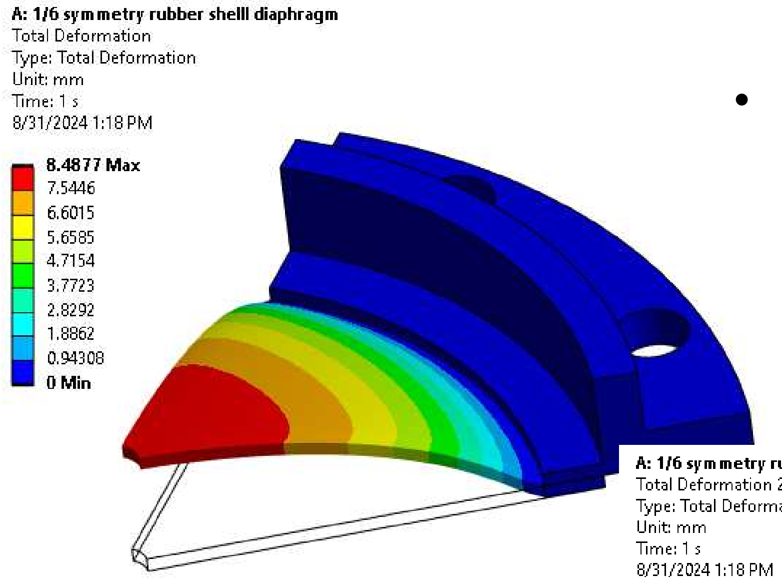
Model 1: Rubber Diaphragm

- Rubber Diaphragm

- The model is fully fixed at the bolt holes as shown and pressure of 0.1 MPa is applied underneath the diaphragm

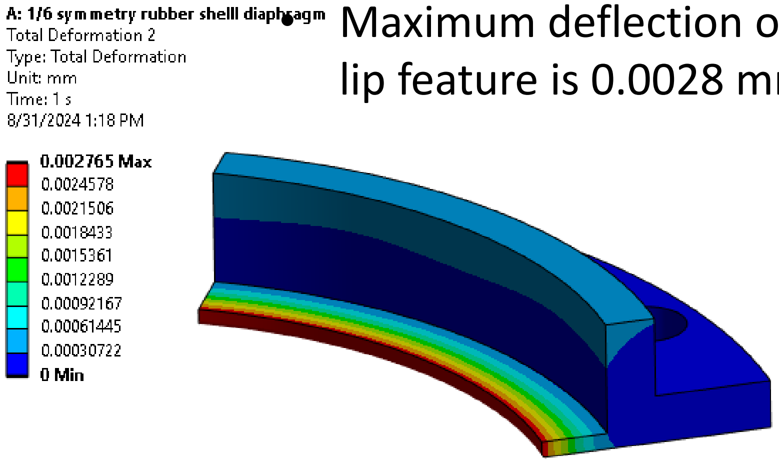






- Total deflection of the diaphragm is ~ 8.5 mm

- But this (below) is the result we're interested in
- Maximum deflection of the lip feature is 0.0028 mm

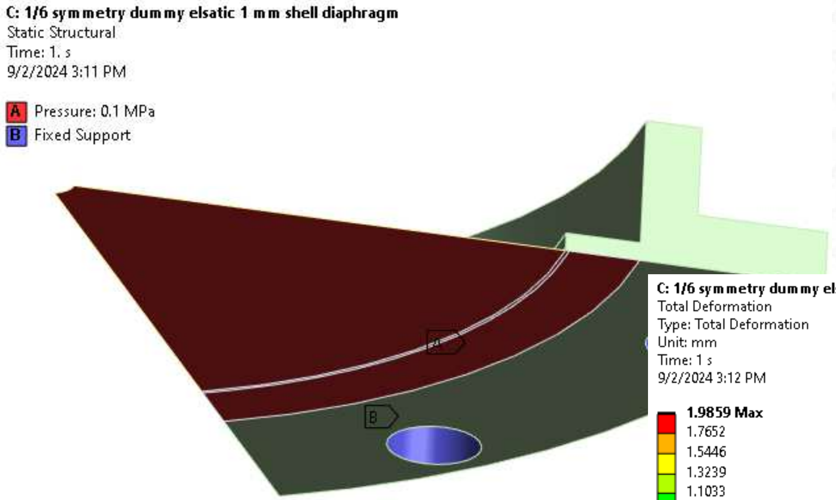


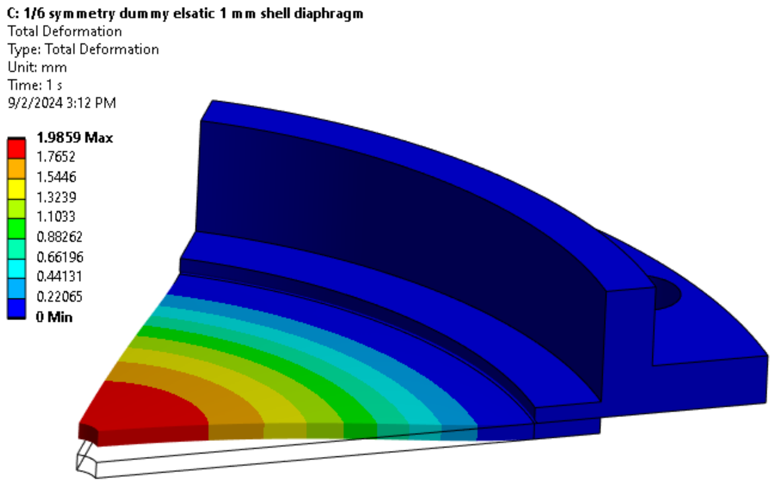
- We next want to build a model of JUST the ratainer (the 'retained'region) and apply a statically equivalent load transmitted by the (excluded) diaphragm

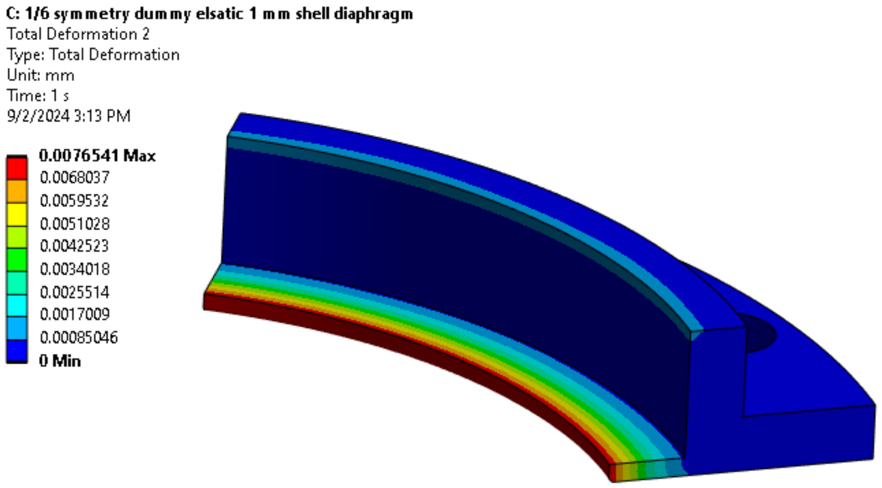
Model 2: Linearly Elastic Diaphragm

- Assuming the diaphragm carries a moment load

- Starting in slide 21, we'll consider how to create a statically equivalent model for a diphgragm which transmits moment loads
- We start by creating a new 1/6-symmery model (we simply duplicate the original in Ansys)
- Then replace the diaphragm material with a linear one. We just create a dummy elastic material whose Young's Modulus is 1/1000 that of steel (recall, that for this approach to be valid, the stiffness of the excluded region must be negligible).
- Again applying the same pressure (0.1 MPa) produces the following new result
- As most readers would suspect, because the linear shell transmits the full moment of the working diaphragm, the resulting displacement is much greater at the steel retainer. The maximum deflection at the lip $\delta = 0.00765\ mm$





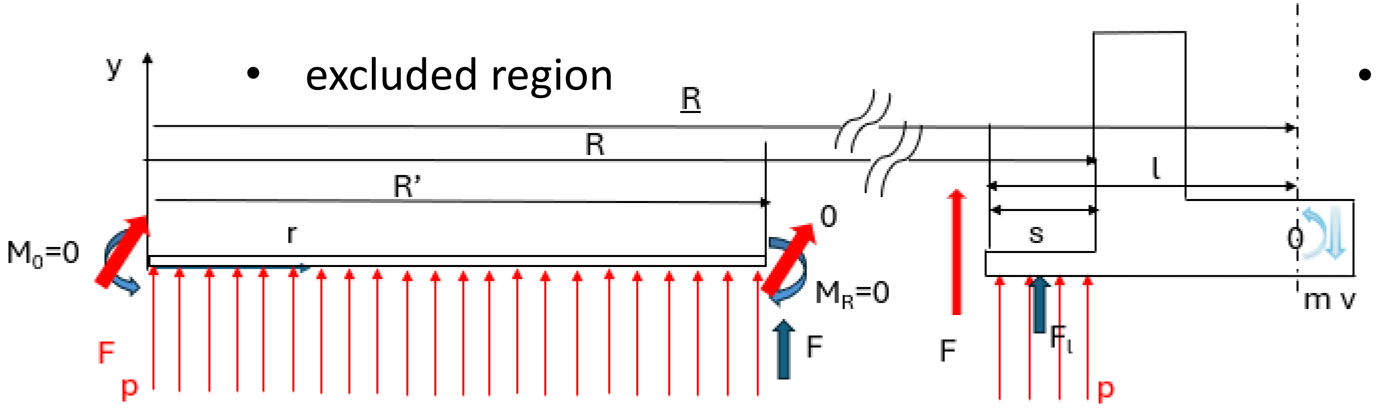


Solution for Model 1: Statically Equivalent Retainer Model

- Assuming the diaphragm carries no moment load

- We'll first assume that, being made of rubber, the working diaphragm can not transmit any moment load
- The solution in this case is quite simple (note that the excluded region is statically determinate –so we dont' need to appeal to any membrane equations for the boundary reactions!)
- Below, F_l is the equivalent load due to the pressure under the lip feature (length s), while F is the transmitted axial load from the working diaphragm
- Moments are summed about the bolt-center radius R.
- Note that if we keep the diaphragm pressure, p under the lip in the retained model, then the only additional calculated force we need to apply to the retained model is $F = p\pi R'^2$ (the equivalent load F_l is represented by the pressure, p. See next slide)

- excluded region
- retained region



$$\sum M = -M_R^{0} - Fl - F_l\left(l - \frac{s}{2}\right) + m = -Fl - F_l\left(l - \frac{s}{2}\right) + m = 0 \quad (10)$$

$$\sum F = F + F_l - v = 0 \quad (11)$$

$$F = p\pi R'^2$$

$$F_l = p\pi (R^2 - R'^2)$$

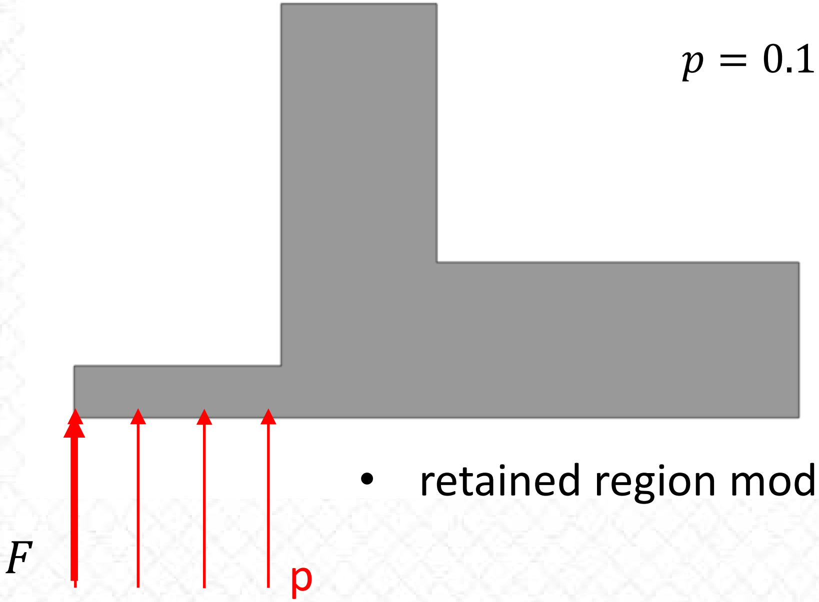
Solution for Model 1: Statically Equivalent Retainer Model

- Assuming the diaphragm carries no moment load: Solution 1

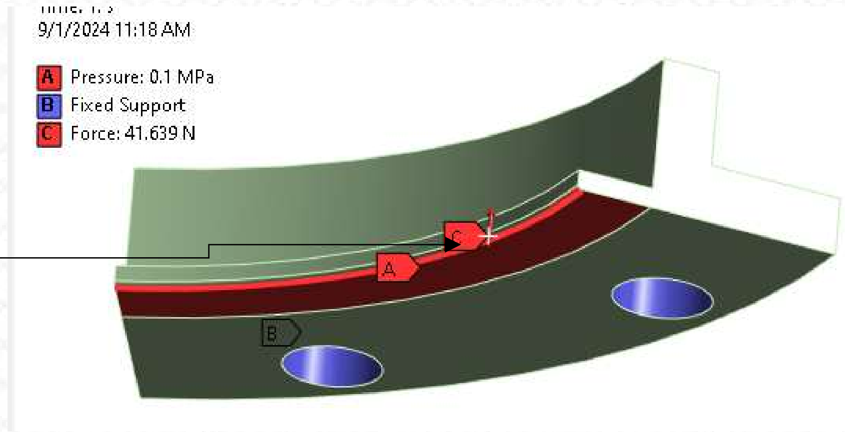
- Substiuting in the model's dimensions to calculate the diaphragm transmitted axial load F of ~249.8 N. But this must be divided by 6 (below) for the 1/6-symmetry model:
- Thus a pressure of 0.1 MPa and 41.64 N are applied as shown below

$$F = \frac{(0.1)(3.14159\ldots)(28.2)^2}{6} = 41.64\ N$$

$$p = 0.1\ MPa$$



- retained region model

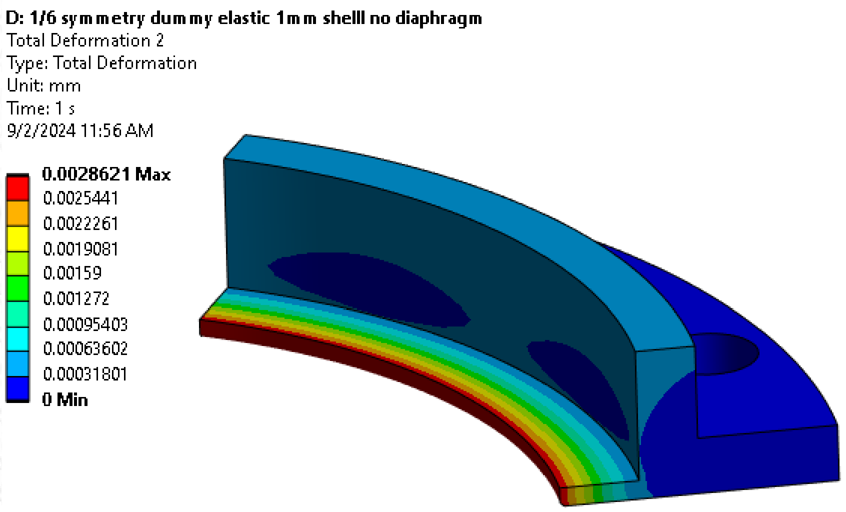


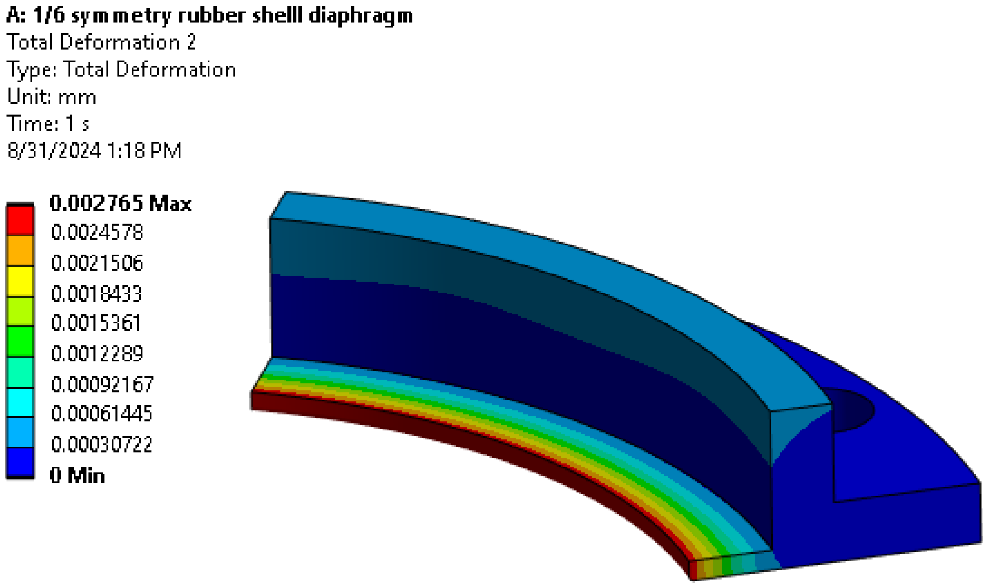
Solution for Model 1: Statically Equivalent Retainer Model

- Assuming the diaphragm carries no moment load: Solution 1

- We can see how well this works in practice below
- The 1/6-symmetry retained model is identical with the retained region of the combined model (slide 15. The retained model was created by simply copying the model of slide 15 and suppressing the diaphragm)

- Compare this solution (below left) to the full model solution (below right)
- Retained region model $\delta = 0.0028625\ mm$
- Full model $\delta = 0.002765\ mm$





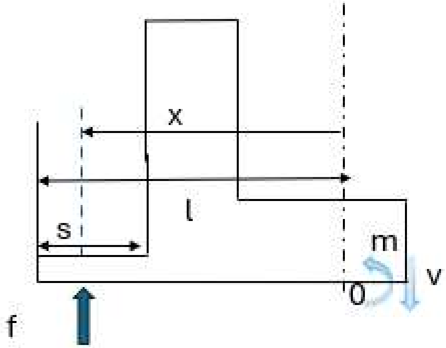
Solution for Model 1: Statically Equivalent Model of Retainer

• Assuming the diaphragm carries no moment load: Solution 2

- Some readers may find the solution of slide 5 and 6 so simple as to be trivial (and it may be)
- However, compare it to the incorrect solution of slide 11 (equation (9))
- Both solutions apply the same load. But by dividing this load by the lip area and applying it as a pressure as on slide 11 , the moment equilibrium of equation (10) is not satisfied. Doing so effectively applies the load F at the location $l - s/2$ (recall that for determinate systems, distributed loads act at their centroid), which will result in a smaller response (not shown. But readers are encouraged to try this with the Workbench model that accompanies this article)

- We may test this idea further by replacing both the diaphragm pressure under the lip AND the transmitted axial load of the excluded region with a single calculated effective axial load, f which satisfies both force and moment equilibrium

- l is the distance from the fixed bolt pattern centers to the boundary of the excluded region



$$\sum M = -fx + m = 0 \quad (12)$$

$$\sum F = f - v = 0 \quad (13)$$

$$fx = Fl + F_l\left(l - \frac{s}{2}\right) \quad (14)$$

- subsituting (10) in to (12) and rearranging

$$(F + F_l)x = Fl + F_l\left(l - \frac{s}{2}\right) \quad (15)$$

- substituting (11) into (14)

$$x = \frac{(F + F_l)l - F_l\frac{s}{2}}{(F + F_l)} \quad (16)$$

- solving (15) for x

$$F = p\pi R'^2$$

$$F_l = p\pi(R^2 - R'^2)$$

$$f = F + F_l$$

Solution for Model 1: Statically Equivalent Model of Retainer

- Assuming the diaphragm carries no moment load: Solution 2
- Substituting in model dimensions to the expressions for F, F_l, f, l, x, r:

$$F = p\pi R'^2 = (0.1)(3.14159\ldots)(28.2)^2 = 249.8\ N$$

$$F_l = p\pi\left(R^2 - R'^2\right) = (0.1)(3.14159\ldots)\left(32^2 - 28.2^2\right) = 71.87\ N$$

$$f = F + F_l = 321.7\ N$$

$$l = \underline{R} - R' = 38.5 - 28.2 = 10.3\ mm$$

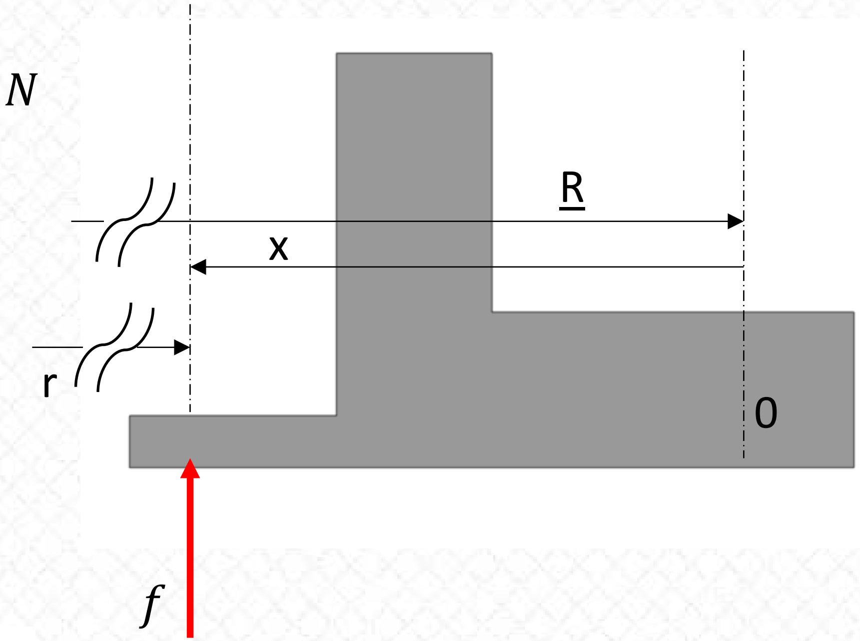
$$s = R - R' = 32 - 28.2 = 3.8\ mm$$

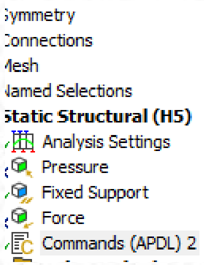
- see slide 12

$$x = \frac{(F+F_l)l - F_l\frac{s}{2}}{(F+F_l)} = \frac{(321.7)(10.3)-(71.87)\frac{3.8}{2}}{321.7} = 9.88\ mm$$

$$r = \underline{R} - x = 38.5 - 9.88 = 28.62\ mm$$

- retained region model



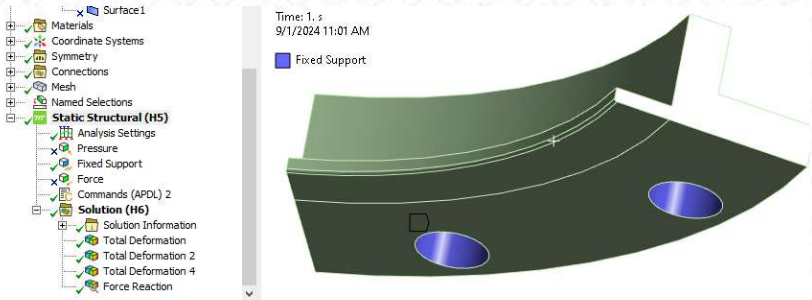


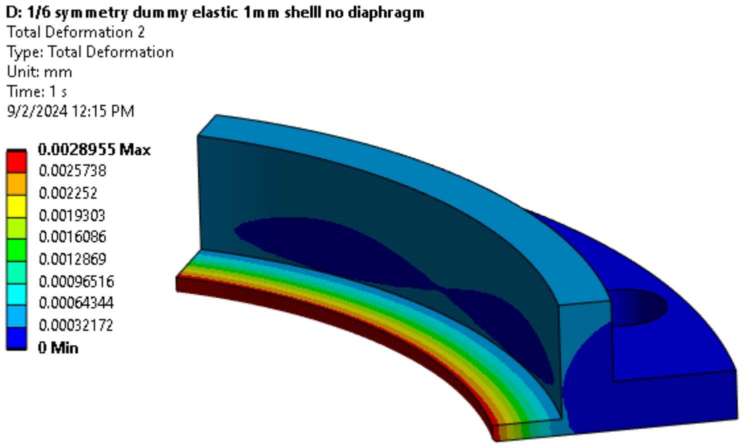

```
/prep7
csys,12                !change to cylindrical csys (identified as csys 12 in WB)
nsel,s,loc,x,28.6245   !select nodes at r=R-x
nsel,r,loc,z,4         !reelect nodes only on bottom surface of retainer
nrotat,all             !transform nodal coordinates to by in cylindrical system
*get,nn,node,,count    !get the number of selected nodes, N
f,all,fz,53.62/nn      !apply requird force f/N on each node
!f,all,my,293.553/nn
allsel,
/solu
```

- Because the load, f is applied at calculated distance r, we apply it with an APDL macro (the load F and pressure p are suppressed)

Solution for Model 1: Statically Equivalent Model of Retainer

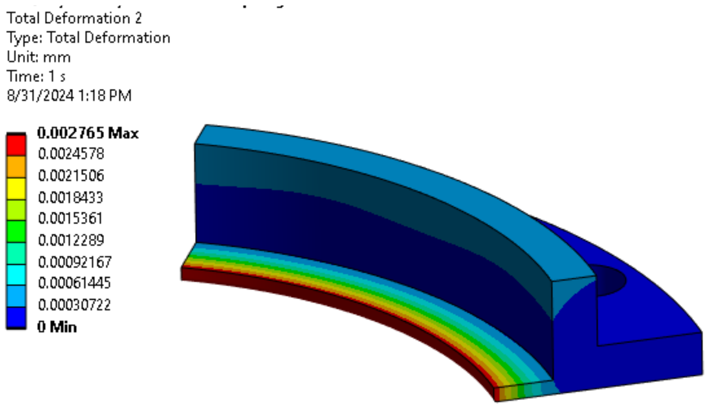
- Assuming the diaphragm carries no moment load: Solution 2
- We can check this new calculated effective force f by replacing the force and pressure applied in the previous model (we suppressed them below) with an APDL macro which applies the load f at the correct radial location, r =R - x





- retained region model with original loads suppressed and macro applied

   - Comparing again (right) to the original model (below right):
   - Retained region model $\delta = 0.002896\ mm$
   - Full model $\delta = 0.002765\ mm$



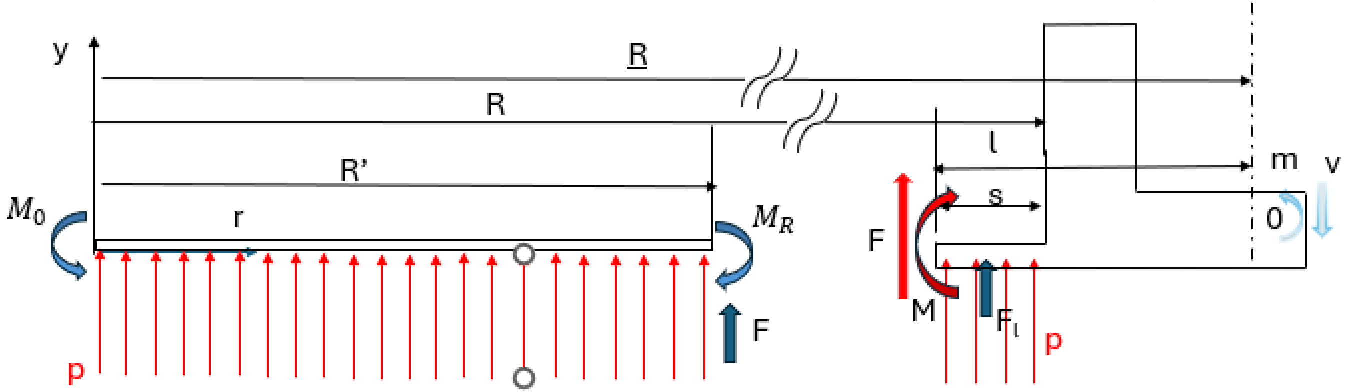
- The increased error has to do with the mesh discretization
- The distance r at which the load is applied must have nodes at that location. The macro attempts to find nodes nearest that location (there is no node at r=28.62 mm). Increasing the mesh density will increase the accuracy here

Solution for Model 2: Statically Equivalent Retainer Model

- Assuming the diaphragm carries a moment load

- Now, suppose that the diaphragm material is not made of an elastomer (or that it operates at a low temperature –below its glass transition)
- In that case, the working diagphragm can be expected to transmit a moment load to the retained region, so the free-body diagram of slide 14 must be modified as below
- This is probably a good time to remind readers that there are two related free-body diagrams here: The excluded region on the left, and the retained region on the right.
- Before, when when the exluded region couldn't transmit moments, the two regions were statically determinate
- But now, the excluded region is statically indeterminate (in the same way that the fixed-fixed beam of slides 8 and 9 are statically indeterminate)!
- Also, we can't use a beam solution to determinate the transmitted moment, M_R(see slide 10)
- We have to turn to plate theory!

- excluded region
- retained region



Solution for Model 2: Statically Equivalent Retainer Model

- Assuming the diaphragm carries a moment load

- Because plate theory is a bit beyond the scope of this article, we'll just point readers to our source (p 166 – 167) and provide key result we'll need below

$$\delta(r) = -\frac{p}{64D}\left(r^2 - R'^2\right)^2$$

- The transverse deflection of the plate

where D is the Modulus of Flexural Rigidity given by:

$$D = \frac{Et^3}{12(1-\nu^2)}$$

- t is the plate thickness

$$M_r = \frac{1}{16}p\left(R'^2(1+\nu) - (3+\nu)r^2\right) \quad (17)$$

- The plate circumferential moment (per unit length)

$$M_\theta = \frac{1}{16}p\left(R'^2(1+\nu) - (1+3\nu)r^2\right) \quad (18)$$

- The plate radial moment (per unit length)

- It is important to note here that the direction indices above use a convention that is opposite to that used in Ansys (and in most discussions of vectors)
- Thus, the moment M_r refers to the moment vector in the θ direction because the 'r' refers to the corresponding strain direction (analogous to $M = -EI\frac{d^2v}{dx^2}$ from beam theory. This vector points 'out of the plane')

Solution for Model 2: Statically Equivalent Retainer Model

- Assuming the diaphragm carries a moment load

- Now, we can use equations (17) and (18) to determine the transferred moments at r=R':

$$M_r = -\frac{pR'^2}{8} \qquad (19)$$

- The plate circumferential moment at r=R' (per unit length)

$$M_\theta = -\frac{p\nu R'^2}{8} \qquad (20)$$

- The plate radial moment at r=R' (per unit length)

  - Since these are provided on a per unit length basis, we can convert them to a full 360^0 basis by multiplying them by $2\pi R'$:

$$M_r = -\frac{p\pi R'^3}{4} \qquad (21)$$

- The plate circumferential moment at r=R'

$$M_\theta = -\frac{p\pi\nu R'^3}{4} \qquad (22)$$

- The plate radial moment at r=R'

  - Finally, we'll note that the plate's transverse shear per unit length, V is given by:

$$V_r = -\frac{pr}{2} \qquad (23)$$

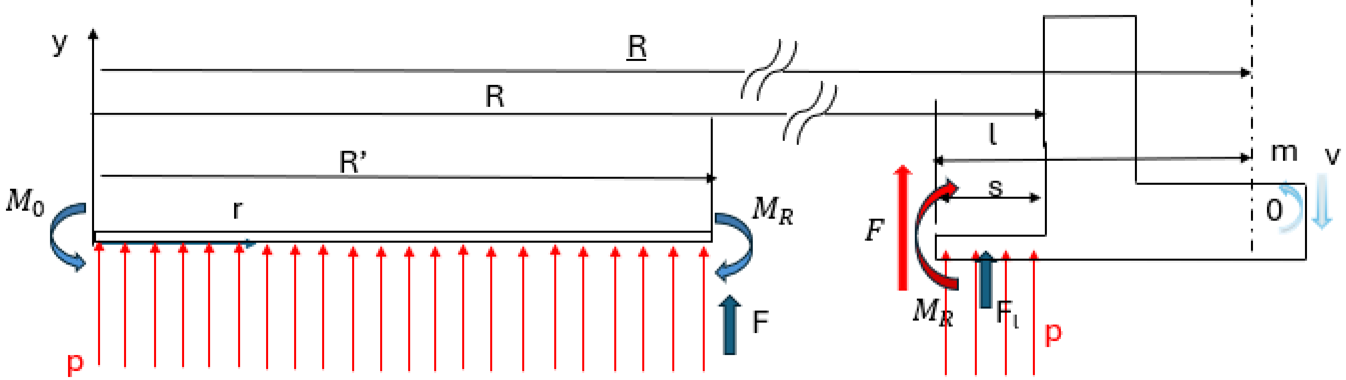
- The plate transverse shear per unit length

Solution for Model 2: Statically Equivalent Retainer Model

- Assuming the diaphragm carries a moment load

- Equation (23) assures us that the diaphragm transmitted axial load F is the same as before (by using equation (23) to calculate the shear at r=R', we again obtain $F = p\pi R'^2$ on a full 360^0 basis)
- So, we can now avoid the mistake of slides 8 and 9 by substituting in the values of F and M_R calculated from plate theory into the free-body diagram:

- excluded region
- retained region



$$\sum M = -M_R - Fl - F_l\left(l - \frac{s}{2}\right) + m = 0 \qquad (24)$$

$$\sum F = F + F_l - v = 0 \qquad (25)$$

$$M_R = \frac{p\pi R'^3}{4}$$

$$F = p\pi R'^2$$

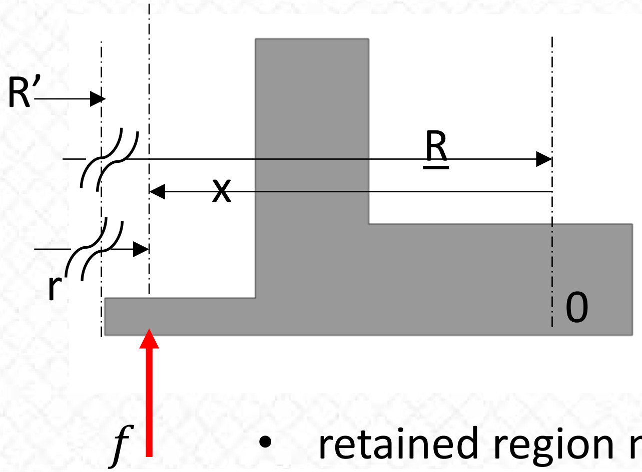
$$F_l = p\pi(R^2 - R'^2)$$

Solution for Model 2: Statically Equivalent Retainer Model

- Assuming the diaphragm carries a moment load: Solution 3

- Now, we can't replace the pressure, p and force F with an equivalent load, f as was done on slides 18 – 20 (solution 2)
- The reason is that equation (16) on slide 17 becomes:

$$x = \frac{\frac{p\pi R'^3}{4} + Fl + F_l\left(l - \frac{s}{2}\right)}{(F + F_l)} \qquad (26)$$



- retained region model

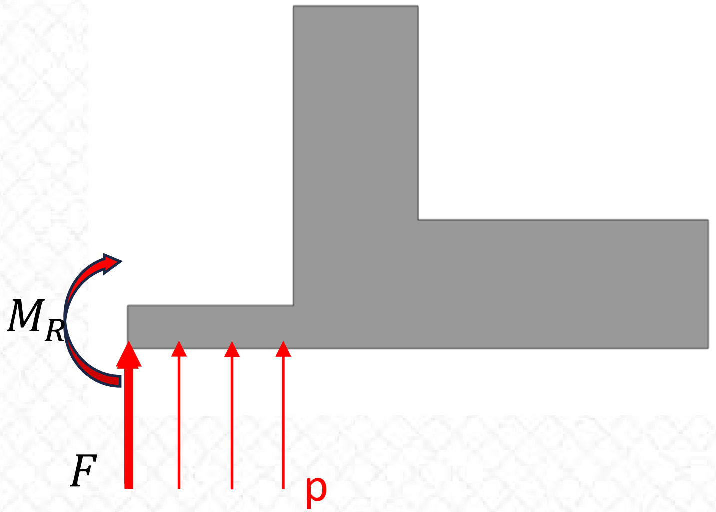
- Making all the substitutions (see slide 19) results in:

$$x = 10.788\ mm$$

- or:

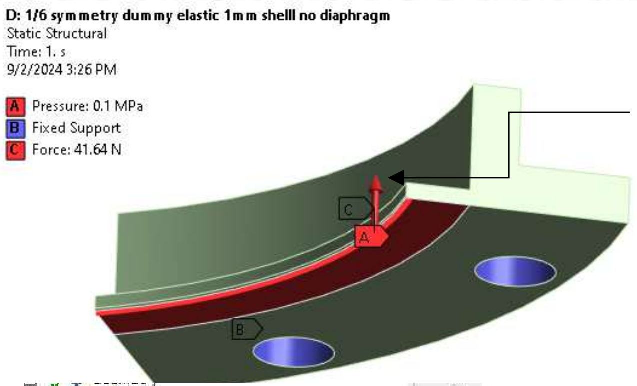
$$r = 38.5 - 10.788 = 27.711\ mm$$

- In other words, $r < R'$ and so is no longer within the excluded region
- This means we have to retain F and p, and find a way to additionally apply the moment M_R as in the figure at right



Solution for Model 2: Statically Equivalent Retainer Model

- Assuming the diaphragm carries a moment load: Solution 3a
  - To create the retained region model, we proceed as before, and simply duplicate the model of slide 25 and suppress the working diaphgragm
  - And again, we use a macro to apply JUST the moment load (keeping in mind what we learned on slide 24)
  - Note that the macro relies on the rotational degrees of freedom of the shell elements representing the retained diaphragm region in order to apply a moment load directly on a single circle of nodes in the cylindrical coordinate system (recalling also the sign convention: M_R should be applied in the θ direction, while M_θ should be applied in the r-direction)

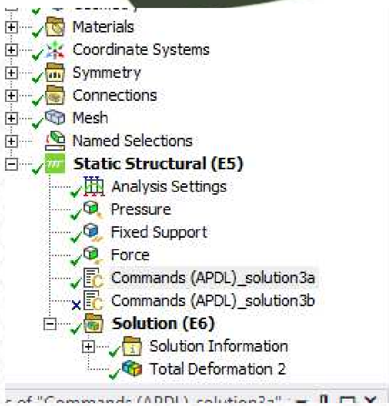


$F/6$

$$M_R = \frac{(0.1)(3.14159\ldots)(28.2)^3}{4*6} = 293.55\ Nmm$$

$$M_\theta = \frac{(0.1)(3.14159\ldots)(0.3)(28.2)^3}{4*6} = 88.066\ Nmm$$

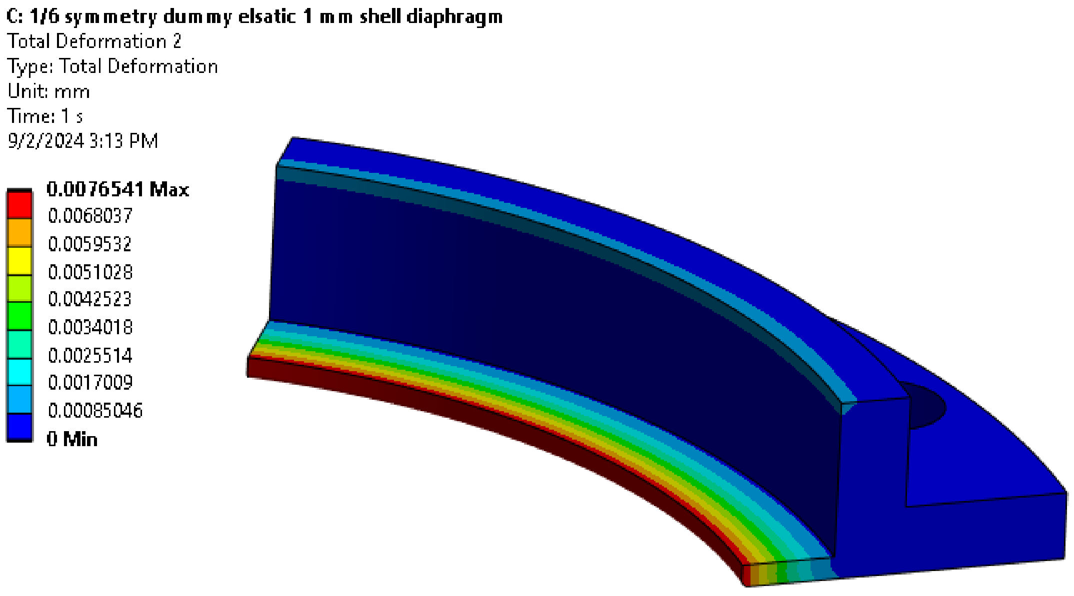
$$p = 0.1\ MPa$$

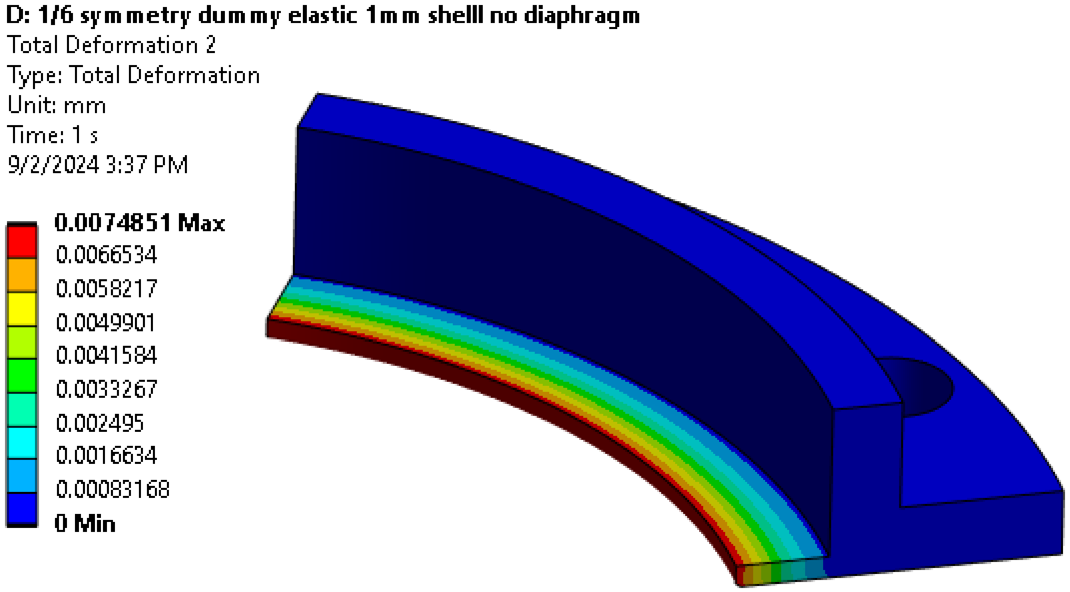



```

/prep7
cmsel,s,membrane
allsel,below,elem
csys,12                    !change to cylindrical csys (identified as csys 12 in WB)
nrotat,all                 !transform nodal coordinates to by in cylindrical system
tr = 0.1                 !selection tolerence
nsel,r,loc,x,28.6245-tr,28.6245+tr     !select nodes at r=R-x
nsel,r,loc,z,4             !reselect nodes only on bottom surface of retainer
*get,nn,node,,count        !get the number of selected nodes, N
f,all,my,293.553/nn        !apply the required M in hoop direction on each node
f,all,mx,88.066/nn         !apply ht required M in r direction on each node
allsel,
/solu

```

Solution for Model 2: Statically Equivalent Retainer Model

- Assuming the diaphragm carries a moment load: Solution 3a
  - Comparing the maximum deflection of the lip (below right) to that of the full model (below left):

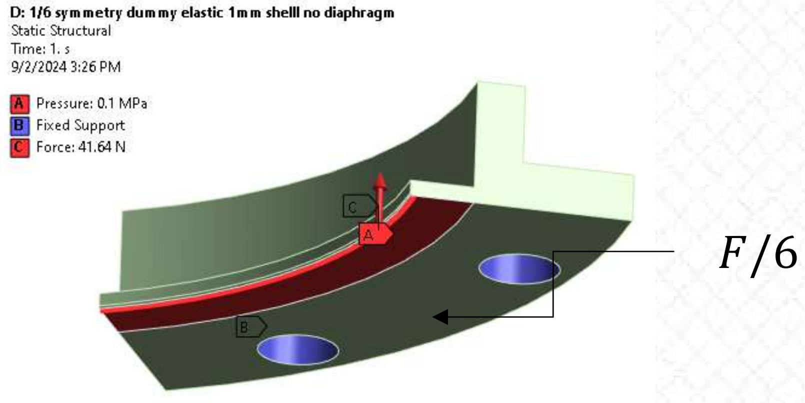


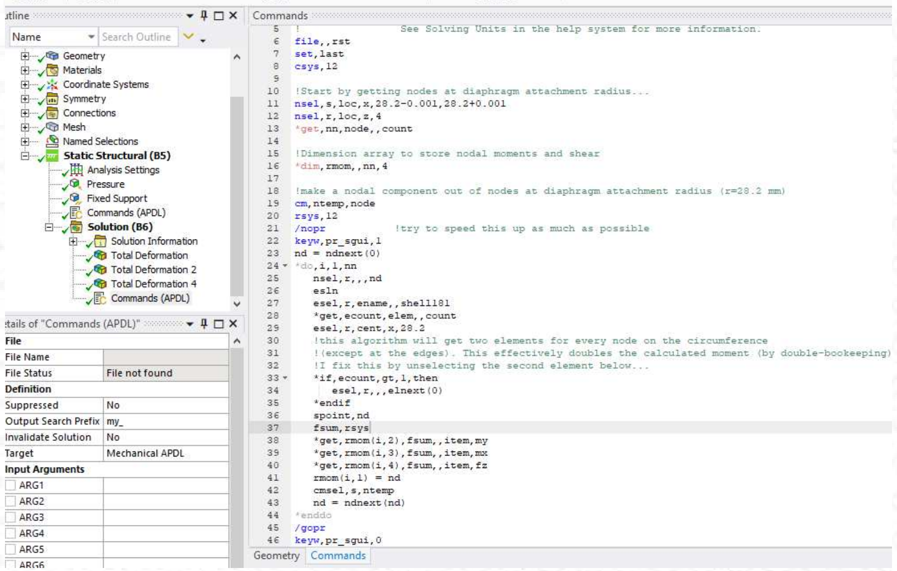


- Retained region model $\delta = 0.007485\ mm$
- Full model $\delta = 0.007654\ mm$

Solution for Model 2: Statically Equivalent Retainer Model

- Assuming the diaphragm carries a moment load: Solution 3b
  - One ‘check’ of the model of solution 3a is to apply the moment reactions obtained by the post-processing macro of model 2 (shown below right)





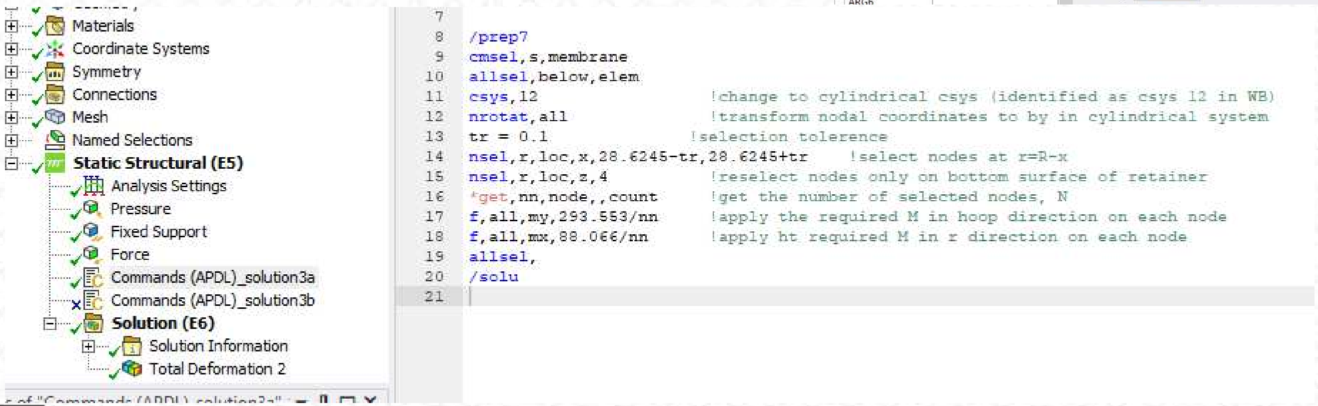
$M_R = 280.21\ Nmm$

$M_\theta = 32.67\ Nmm$

$p = 0.1\ MPa$

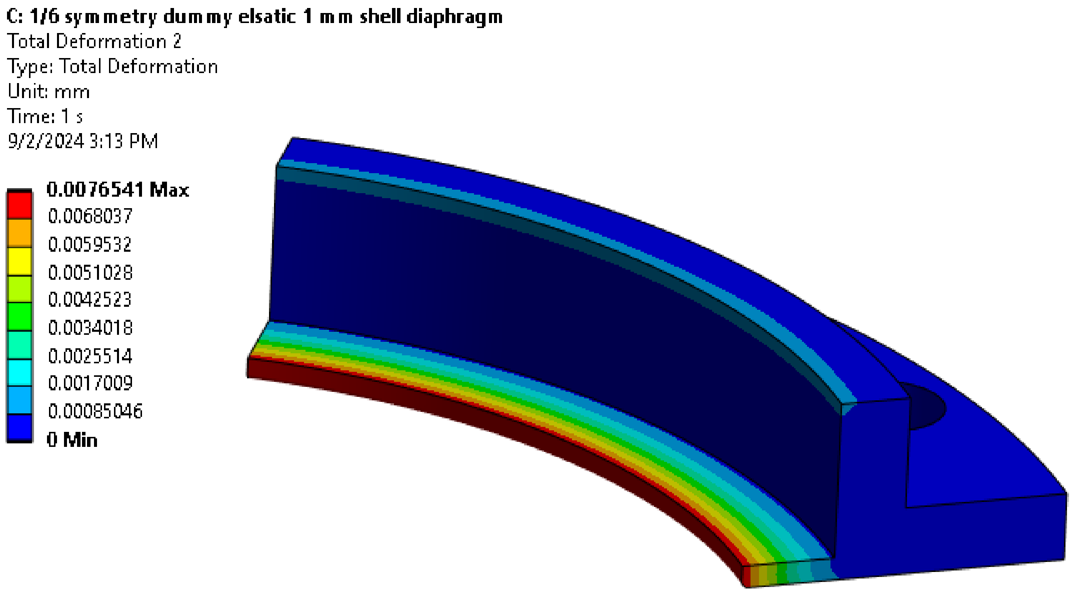
Results

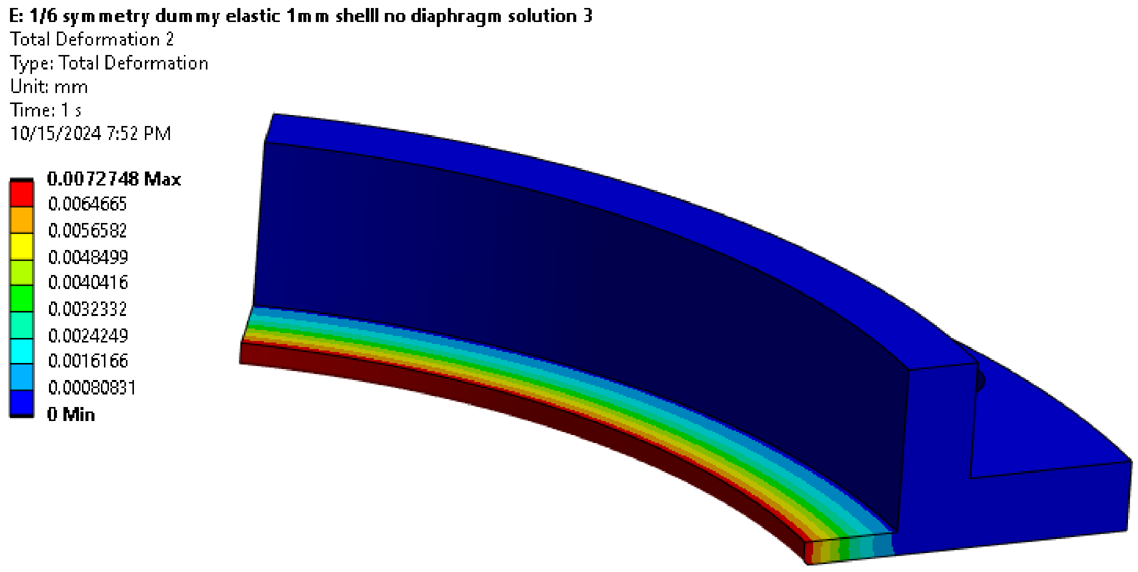
my_mx	32.67
my_my	280.21
my_fz	42.078



Solution for Model 2: Statically Equivalent Retainer Model

- Assuming the diaphragm carries a moment load: Solution 3b
  - Comparing the maximum deflection of the lip (below right) to that of the full model (below left):





- Retained region model $\delta = 0.00727\ mm$
- Full model $\delta = 0.007654\ mm$

- The main reason for this discrepancy is the mesh resolution of the full model (model2) from which the moment reactions were extracted. It differs from the theoretical result by 4.5 percent (280.21 MPa vs 293.55 MPa over the 1/6 sector)

Solution Summary: Statically Equivalent Retainer Model

- The table below summarizes all the estimates in this article
- Thanks to Pablo Alvarez of Sol Aero for his suggestion of using Roark's Formulas for Stress and Strain for more verifcation of the retained model deflection

Model	Transverse Load (N: 1/6 sector basis)	Moment Load (N-mm: 1/6 sector basis)	Estimated Transverse Deflection (mm)	Notes
Full (diaphragm + retainer) 1/6 symmetry: Rubber Diaphragm (no moment transfer)	41.6	None	0.002765	model 1: Slide 14. Transvers shear and pressure only. Workbench system A
Full (diaphragm + retainer) 1/6 symmetry: Dummy Linear Diaphragm (no moment transfer)	41.6	293.55	0.00765	model 2: Slide 15. Tramsverse shear, pressure, and moment loading. Workbench system B
Retained region 1/6 symmetry (no moment transfer)	41.6	None	0.00286	solution 1. Slide 18. Transverse shear and pressure only. Workbench system C
Retained region 1/6 symmetry (no moment transfer)	41.6	None	0.00289	solution 2. Slide 21. Transverse shear and pressure only. Workbench system D
Retained region 1/6 symmetry (includes moment transfer)	41.6	293.55	0.00749	solution 3a. Slide 28. Transverse shear, pressure, and moment loading. Workbench system E
Retained region 1/6 symmetry (includes moment transfer)	42	280.21	0.00727	solution 3b (loads calculated by macro).Transverse shear, pressure, and moment loading. Slide 30. Workbench
Retained region full 360 (Roark's estimate)	41.6	293.55	0.00726	Appendix B: Transverse shear, pressure, and moment loading. Roark's Estimate

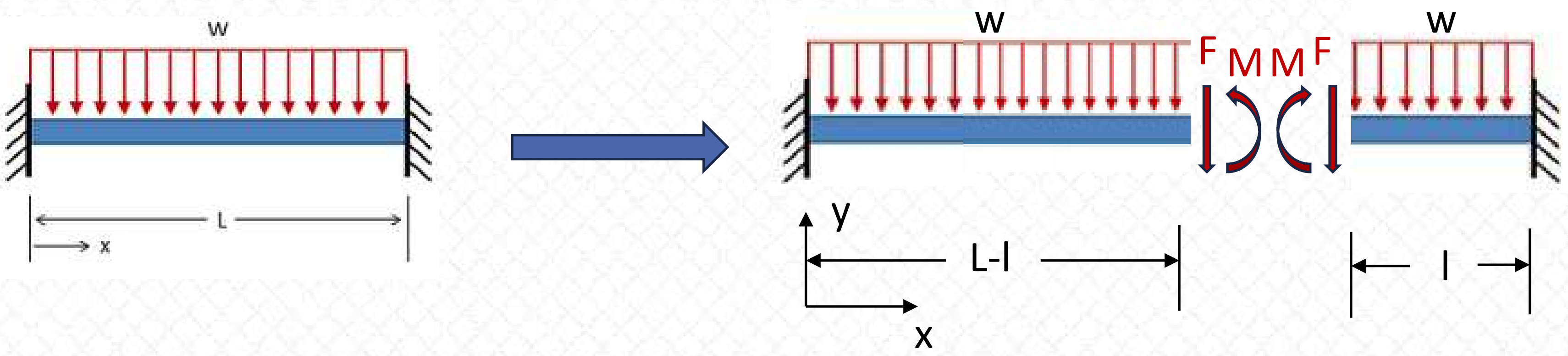
Summary

- In this article, we looked at how to determine statically equivalent loading of a region within a structural finite element model in such a way that it could be analyzed in isolation
- We first discussed the preconditions for doing this in general (see slides 3- 7). We learned, for example, that this can't be done in situations where the excluded region's stiffness contribution can't be ignored. In those situations, engineers should consider submodeling or substructuring.
- We also discussed common pitfalls engineers face when performing moment and force equilibrium calculations for finite element models (see slides 8 – 11). Perhaps the most common mistake engineers make is treating statically indeteminate models as though they are in fact statically determinate. We point out that the working diaphragm model we wish to exclude is statically indeterminate *if it carries a moment load* (see slide 21), and we overcame this limitation by solving the associated differential plate equation for those diaphragms that carry moment loads.
- In the end, we consider two kinds of working diaphragm (model 1 and 2 on slides 13 and 14): ones that don't carry moment loads (made of rubber, for example), and ones that do.
- We offer two statically equivalent retained region solutions for the first case (solutions 1 and 2), and a third solution for second case (solution 3)

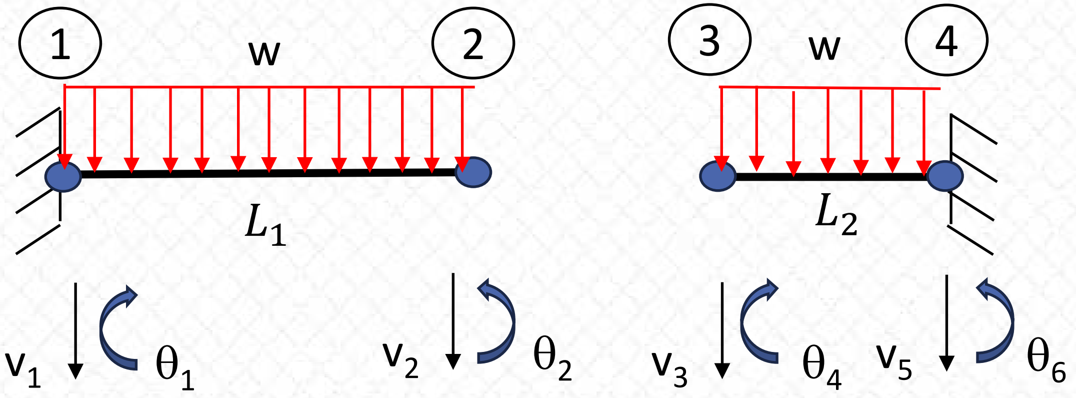
Appendix

Appendix A

Modeling a Fixed-Fixed beam as two coupled cantilevers



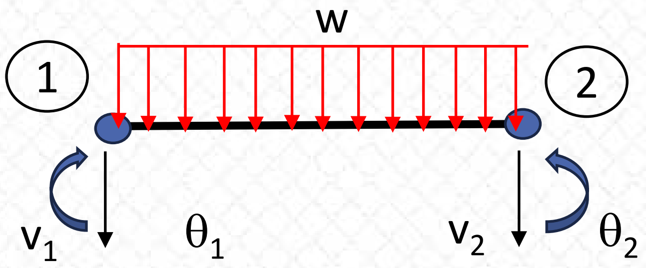
- We want to model the fixed-fixed beam as two coupled cantilevers as shown above
- To do this, we'll couple the two beams using the Lagrange Muliplier approach (to explicitly extract the coupling force and moment)
- We'll use two classical beam elements as shown below



Appendix A

Modeling a Fixed-Fixed beam as two coupled cantilevers

- Each beam is characterized by the degrees of freedom as shown below and matrices **k**, **u**, and **f** as described here



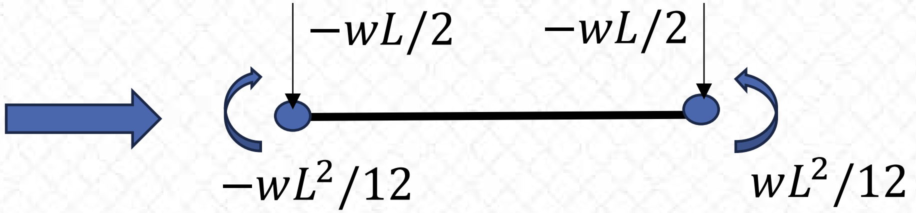
$$\mathbf{k} = \frac{EI}{L^3}\begin{pmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{pmatrix}$$

$$\mathbf{u} = [v_1 \quad \theta_1 \quad v_2 \quad \theta_2]^T$$

$$\mathbf{f} = [F_1 \quad M_1 \quad F_2 \quad M_2]^T$$

- The forces and moments may be obtained from the distributed load, w as shown below (see slide 31 of this reference):
- Note that the equations produce an effective negative bending moment for positive w (but w is negative in the figures we've been using, so we make the adjustments shown)

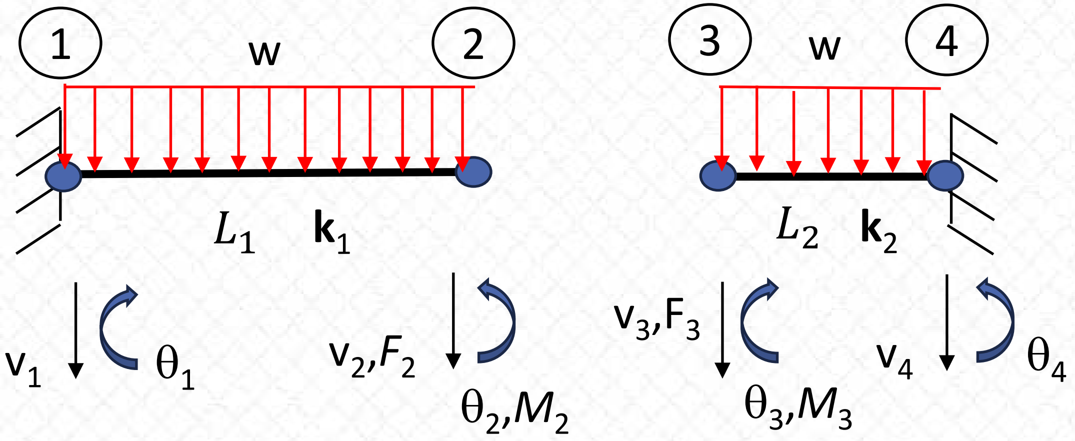
$$\mathbf{f} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \begin{Bmatrix} -wL/2 \\ -wL^2/12 \\ -wL/2 \\ wL^2/12 \end{Bmatrix}$$



Appendix A

Modeling a Fixed-Fixed beam as two coupled cantilevers

- Next, we note that all degrees of freedom are fixed at nodes 1 and 4 (meaning that $v_1 = \theta_1 = v_4 = \theta_4 = 0$) leaving us with two the 2 x 2 system matrices shown below



$$\mathbf{k}_1 = \frac{EI}{L_1^3}\begin{pmatrix} 12 & -6L_1 \\ -6L_1 & 4L_1^2 \end{pmatrix} \qquad \mathbf{k}_2 = \frac{EI}{L_2^3}\begin{pmatrix} 12 & 6L_2 \\ 6L_2 & 4L_2^2 \end{pmatrix}$$

$$\mathbf{u}_1 = [v_2 \quad \theta_2]^T \qquad \mathbf{u}_2 = [v_3 \quad \theta_3]^T$$

$$\mathbf{f}_1 = [-wL/2 \quad wL^2/12]^T \qquad \mathbf{f}_2 = [-wL/2 \quad -wL^2/12]^T$$

Appendix A

Modeling a Fixed-Fixed beam as two coupled cantilevers

- We want to couple the two system matrices by coupling the degrees of freedom such that $\mathbf{u}_1 = \mathbf{u}_2$ and introducing the Lagrange mulipliler, $\boldsymbol{\lambda}$ and coupling matrix **c**:

$$\mathbf{k}_1 \cdot \mathbf{u}_1 + \mathbf{c}_1 \cdot \boldsymbol{\lambda}_1 = \mathbf{f}_1$$
$$\mathbf{k}_2 \cdot \mathbf{u}_2 + \mathbf{c}_2 \cdot \boldsymbol{\lambda}_2 = \mathbf{f}_2$$
$$\mathbf{c} \cdot \mathbf{u} = \mathbf{0}$$

- where $\boldsymbol{\lambda} = [F \quad M]^T$
- and the coupling matrix **c** is determined as follows...:
- The degrees of freedom are coupled by imposing:

$$v_2 - v_3 = 0 \rightarrow [1 \quad 0 \quad -1 \quad 0]\begin{Bmatrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

- And:

$$\theta_2 - \theta_3 = 0 \rightarrow [0 \quad 1 \quad 0 \quad -1]\begin{Bmatrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Appendix A

Modeling a Fixed-Fixed beam as two coupled cantilevers

- The DoF coupling can be compactly represented with coupling matrix, **c**:

$$\mathbf{c} \cdot \mathbf{u} = \mathbf{d} = \mathbf{0}$$

- where

$$\mathbf{c} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

- Finally, the coupled system is written as:

$$\begin{pmatrix} \mathbf{k} & \mathbf{c}^T \\ \mathbf{c} & \mathbf{0} \end{pmatrix} \begin{Bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f} \\ \mathbf{0} \end{Bmatrix}$$

- Note that this is a 6 x 6 system of equations –ready to be solved

- where:

$$\mathbf{k} = \begin{pmatrix} \mathbf{k}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{k}_2 \end{pmatrix}$$

$$\mathbf{u} = [\mathbf{u}_1 \quad \mathbf{u}_2]^T$$

$$\mathbf{f} = [\mathbf{f}_1 \quad \mathbf{f}_2]^T$$

Appendix A

Modeling a Fixed-Fixed beam as two coupled cantilevers

- Solving this system in Mathematica results in:

$$v_2 = \frac{l^2 L w (l-L)^2 \left(\mathrm{E}_1 l^2 (2l-3L) - \mathrm{E}_2 (l-L)^2 (2l+L)\right)}{24 \mathrm{I}_1 \left(\mathrm{E}_1^2 l^4 - 2\mathrm{E}_1\mathrm{E}_2 l (l-L)\left(l^2 - lL + 2L^2\right) + \mathrm{E}_2^2 (l-L)^4\right)}$$

$$\theta_2 = \frac{l L w (l-L) \left(\mathrm{E}_2 (l-L)^3 (l+L) - \mathrm{E}_1 l^3 (l-2L)\right)}{12 \mathrm{I}_1 \left(\mathrm{E}_1^2 l^4 - 2\mathrm{E}_1\mathrm{E}_2 l (l-L)\left(l^2 - lL + 2L^2\right) + \mathrm{E}_2^2 (l-L)^4\right)}$$

$$v_3 = \frac{l^2 L w (l-L)^2 \left(\mathrm{E}_1 l^2 (2l-3L) - \mathrm{E}_2 (l-L)^2 (2l+L)\right)}{24 \mathrm{I}_1 \left(\mathrm{E}_1^2 l^4 - 2\mathrm{E}_1\mathrm{E}_2 l (l-L)\left(l^2 - lL + 2L^2\right) + \mathrm{E}_2^2 (l-L)^4\right)}$$

$$\theta_3 = \frac{l L w (l-L) \left(\mathrm{E}_2 (l-L)^3 (l+L) - \mathrm{E}_1 l^3 (l-2L)\right)}{12 \mathrm{I}_1 \left(\mathrm{E}_1^2 l^4 - 2\mathrm{E}_1\mathrm{E}_2 l (l-L)\left(l^2 - lL + 2L^2\right) + \mathrm{E}_2^2 (l-L)^4\right)}$$

$$F = \frac{w \left(\mathrm{E}_1^2 l^5 - \mathrm{E}_1\mathrm{E}_2 l (l-L)(2l-L)\left(l^2 - lL + 3L^2\right) + \mathrm{E}_2^2 (l-L)^5\right)}{2 \left(\mathrm{E}_1^2 l^4 - 2\mathrm{E}_1\mathrm{E}_2 l (l-L)\left(l^2 - lL + 2L^2\right) + \mathrm{E}_2^2 (l-L)^4\right)} \quad \text{(A1)}$$

$$M = \frac{w \left(\mathrm{E}_1^2 l^6 - \mathrm{E}_1\mathrm{E}_2 l^2 (l-L)^2 \left(2l^2 - 2lL + 9L^2\right) + \mathrm{E}_2^2 (l-L)^6\right)}{12 \left(\mathrm{E}_1^2 l^4 - 2\mathrm{E}_1\mathrm{E}_2 l (l-L)\left(l^2 - lL + 2L^2\right) + \mathrm{E}_2^2 (l-L)^4\right)} \quad \text{(A2)}$$

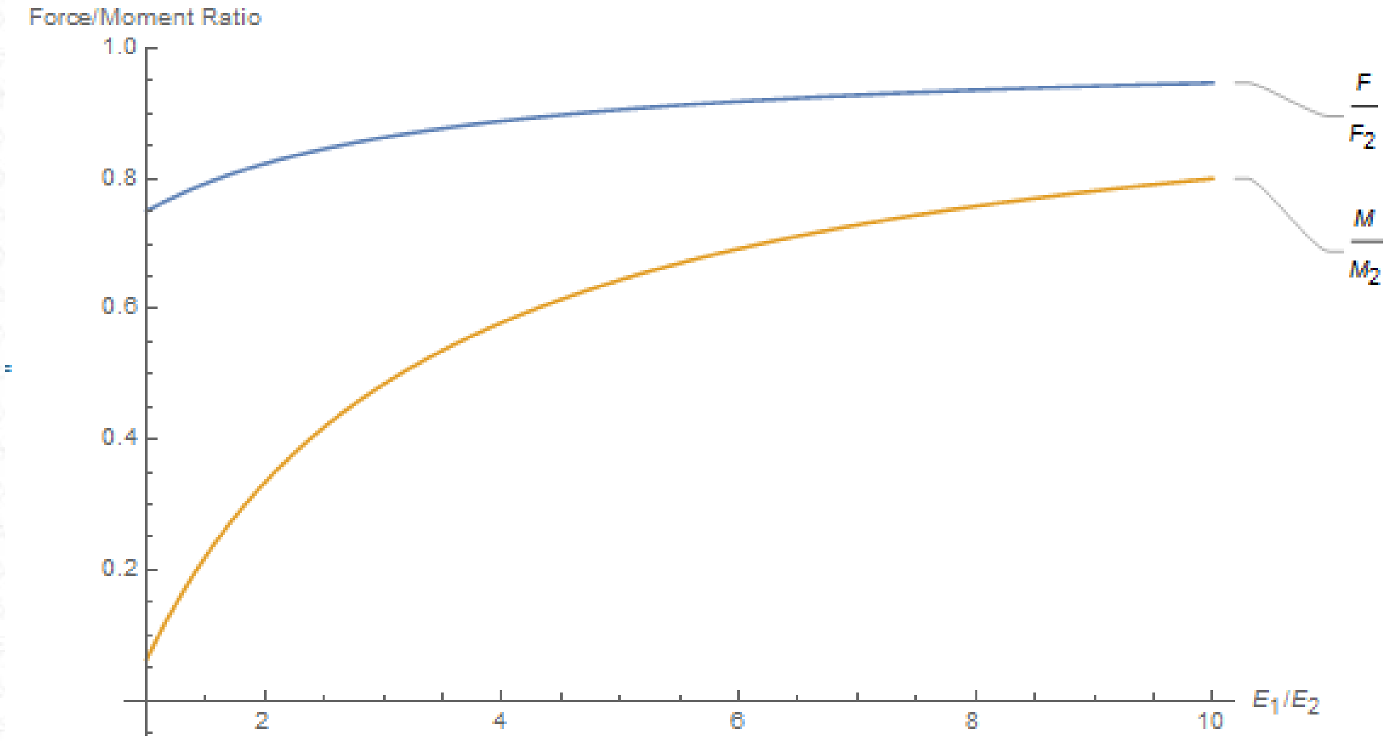
Appendix A

Modeling a Fixed-Fixed beam as two coupled cantilevers

- The equations (A1) and (A2) now provide a convenient estimate for moment and shear force transfer between beams of differing material!
- Furthermore, these equations provide us with a quantitative description of how the Lagrange Multpliers (the coupling force and moment) compare with the shear force and moment of beam 1 ($F_2 = \frac{w\,L_1}{2}, M_2 = wL_1^2/12$)
- This is important because users will naturally want to resort to the approximation $F \approx F_2$ and $M \approx M_2$ when $E_1 \ll E_2$
- When $E_1 = E_2$, we can use Equations (1) and (2) (from slide 5) to obtain:
    - $\frac{F}{F_2} = \frac{3}{4}$
    - $\frac{M}{M_2} = 1/16$
- The graph on the next slide uses (A1) and (A2) to show the ratios F/F_2 and M/M_2 as functions of E_1/E_2

Appendix A

Modeling a Fixed-Fixed beam as two coupled cantilevers



- Finally, setting $E_1 = E_2$ transforms the coupled-beam solution of slide 37 to:

$$\begin{Bmatrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \\ F \\ M \end{Bmatrix} = \begin{Bmatrix} -\dfrac{wl^2(l-L)^2}{24E_1I_1} \\ \dfrac{wl(l-L)(2l-L)}{12E_1I_1} \\ -\dfrac{wl^2(l-L)^2}{24E_1I_1} \\ \dfrac{wl(l-L)(2l-L)}{12E_1I_1} \\ \dfrac{w}{2}(2l-L) \\ \dfrac{1}{12}w\left(6l^2-6lL+L^2\right) \end{Bmatrix}$$

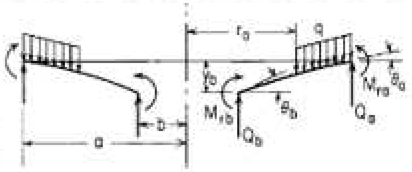
- Thus reproducing the original solution on slide 6

Appendix B

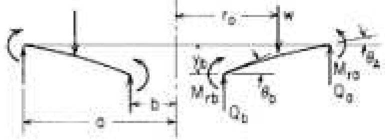
Annulus Deflection Estimate (Roark’s Formulas for Stress and Strain)

- This article focused on estimating the deflection of the retained region of a structure using finite element analysis
- However, the retained region of interest is a simple annulus.
- Because of this, we could also resort to closed-form analytical estimates for such a structure (the axisymmetric plate solution we reference on slide 23 also easily admits solution of plates with a hole –but we’d have to solve for some constants)
- As reader Pablo Alvarez of Sol Aero reminded us, one could also just ‘look it up’ in one of many convenient tables of such solutions. We’d like to do that here to compare our deflection estimates to those found by using tables.
- We’ll use Roark's Formulas for Stress and Strain -Seventh Edition
- We’ll have to superpose the three linear solutions below

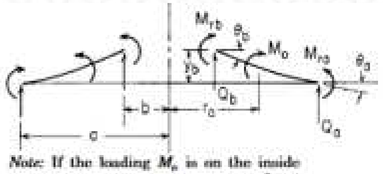
- annulus with a pressure load



- annulus with a line load



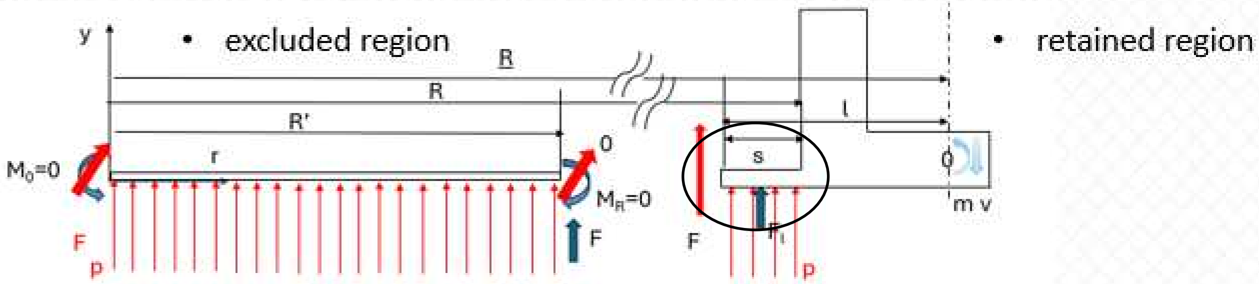
- annulus with a moment load



Appendix B

Annulus Deflection Estimate (Roark's Formulas for Stress and Strain)

- We further have to look for our boundary conditions (fixed outer radius, free inner radius). We want to estimte the deflection of the 'lip' (flange) feature, assuming that it's fixed at r=R'
- We find the following three expressions for the transverse deflection, y:



- moment load:

$$y = K_y \frac{M_r a^2}{D}$$

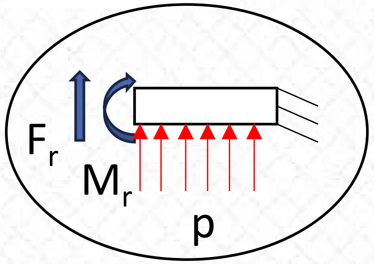
- line load:

$$y = K_y \frac{V_r a^3}{D}$$

- pressure load:

$$y = K_y \frac{q a^4}{D}$$

- D is the flexural modulus (see slide 23), K_y is the transverse deflection stiffness (different for each case as shown on the next slide)
- V_r, M_r are the transverse line load and circumferential bending moment respectively
- a is the radius to the base of the flange (R')



Appendix B

Annulus Deflection Estimate (Roark's Formulas for Stress and Strain)

- Looking up K_y for each of the three cases by interpolating from the corresponding tables and plugging into the three formulas on the previous slide allows us to estimate the total transverse deflection
- The details are captured in the accompanying spreadsheet "roarks_formuas_plate_w_hole.xlsx"

variable	value	description
nu	0.300	Poisson's Ratio
t	1.000	thickness (mm)
D	18315.018	flexural modulus (N mm)
p	0.100	plate pressure (MPa)
R'	28.200	diagphragm attachment radius (mm)
R	32.000	flange base radius (mm)
Vr	1.410	plate transverse shear/unit length (slide 24
Mr	9.941	plate circum. moment/unit length (slide 24

moment load at b				defl. v
ky	0.0048	0.0373	0.88125	
b/a	0.9	0.7	0.007847	-0.004361

line load at b				defl. v
ky	-0.0003	-0.0071	0.88125	
b/a	0.9	0.7	-0.00094	-0.002365

uniform pressure				defl. v
ky	-0.00001	-0.0009	0.88125	
b/a	0.9	0.7	-9.3E-05	-0.000535

Total	-0.007261

- Compare the deflection without moment transfer to values on slides14, 18 and 21
- Compare the total deflection to values on slides 15, 28 and 30